# Notes on Quantum Mechanics

Andrew Forrester  January 28, 2009

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1 Ideas and Questions

1.1 Transfer Quicknotes info into this file

1.2 Ideas

- We start with a knowledge of advanced Classical Mechanics and Electrodynamics and perhaps some basic macroscopic Thermodynamics
- An historical approach takes us through the old quantum theory and then into the newer theory, finally entering “second quantization” (QFT will have its own notes)
- Mathematical approach: Present recipe for moving from Classical Mechanics to Quantum Mechanics, present additional postulates, and develop important applications
- Many philosophical questions to be asked (and various possible answers to be listed)

1.3 Fundamental Questions

- What are the most fundamental postulates?
  Should derive $\langle x | P = -i\hbar \partial_x \langle x |$ and/or $[X, P] = i\hbar$
- Why do we use complex numbers? (Where does the convenience first show up? Look at Sakurai again.)
- Why do we declare momentum to be related to the argument of the wavefunction and position to be related to the magnitude (rather than imaginary part versus real part)?
- What are the relevant ideas from linear algebra, in particular, infinite dimension spaces, various types of Hilbert spaces, and other applicable spaces?
- (With Chris’s simulation of the double-slit experiment in mind...) Could we perform the double-slit experiment assuring that the same amount of energy is taken from the electron each time it is measured, so that coherence is maintained and the diffraction pattern does not disappear?
- Is it possible that wavefunctions do not “collapse”?
- ...

1.4 More Questions

- Can the sea of vacuum fluctuations be considered as a kind of aetherial medium?
- The Schrödinger Equation is like a conservation of energy equation ($\hat{H}\Psi = \hat{E}\Psi$ or $\hat{H}\Psi = i\hbar \frac{d}{dt}\Psi$). Why should interaction terms be placed in the Hamiltonian? Could they be placed on one side of the equation, representing a system with interactions where energy is not conserved? Is there already a convention from Hamiltonian/Lagrangian formalism which provides for interaction terms in the Hamiltonian?)
  When a Hamiltonian has off-diagonal interaction terms, that merely means that the Hamiltonian is not being expressed in terms of eigen-energy states. Diagonalize it and there will be no interaction terms...
- Hey, what’s Hund’s Rule?
  Hund’s Rules: http://hyperphysics.phy-astr.gsu.edu/hbase/atomic/hund.html#c1
- Term Symbols: http://hyperphysics.phy-astr.gsu.edu/hbase/atomic/term.html#c1
- How are the Bohr-Sommerfeld quantizations conditions equivalent to Heisenberg’s quantum mechanics (to first order)?
- When should the “minimal coupling” rule ($p \rightarrow p + eA(r)$) be used, and why is it called that?
Minimal coupling rule for canonical generalized momentum in relativistic notation:

$$\pi_\mu = m \dot{x}_\mu + A_\mu(x, \pi)$$

1.5 Answered or Partially Answered Questions

- Why is the Hamiltonian formalism used but not the Lagrangian formalism? (Lagrangian Formalism → Path-Integral Formalism)

2 The Big Picture

Topics

Subtopics, Supertopics, Subfields, Superfields, Context of Physics (as a whole)...

- Quantum “Statics” (Not really static due to uncertainty principle, which is due to...)
- Quantum Dynamics

Domains

- Classical versus Quantum
- Relativistic versus Non-relativistic

3 Terms and Notation

3.1 Quantity Overview

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{J}$</td>
<td>angular momentum operator</td>
<td>(kg $\cdot$ m$^2$/s)</td>
<td>$nh$ or $n\frac{\hbar}{2}$</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>orbital angular momentum operator</td>
<td>(kg $\cdot$ m$^2$/s)</td>
<td></td>
</tr>
<tr>
<td>$\hat{S}$</td>
<td>spin angular momentum operator</td>
<td>(kg $\cdot$ m$^2$/s)</td>
<td></td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>magnetic dipole moment operator</td>
<td>(A $\cdot$ m$^2$)</td>
<td></td>
</tr>
<tr>
<td>$\gamma_p$</td>
<td>gyromagnetic ratio for particle $p$</td>
<td>(C/kg)</td>
<td>$[\mu/\hat{J}] = (Am^2)/(m\cdot kg\cdot m/s) = (C/kg)$</td>
</tr>
<tr>
<td>$g_p$</td>
<td>g-factor</td>
<td>(dimensionless)</td>
<td></td>
</tr>
<tr>
<td>$\lambda$</td>
<td>helicity</td>
<td>(kg $\cdot$ m$^2$/s)</td>
<td>(angular momentum)</td>
</tr>
</tbody>
</table>

The hat notation $\hat{V}$ denotes an operator (rather than, say, a unit vector).

Some people define the quantum operators $\hat{J}$, $\hat{L}$, and $\hat{S}$ to be dimensionless.
### Dimensionless Quantum Numbers (qn’s)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>principal qn (for hydrogen)</td>
<td>$n \in \mathbb{Z}^+ = {1, 2, 3, \ldots}$</td>
</tr>
<tr>
<td>$N$</td>
<td>radial qn (for hydrogen)</td>
<td></td>
</tr>
<tr>
<td>$j$</td>
<td>total angular momentum qn</td>
<td></td>
</tr>
<tr>
<td>$\ell$</td>
<td>orbital angular momentum qn</td>
<td>$\ell \in {0, 1, 2, \ldots, n - 1}$ (for hydrogen)</td>
</tr>
<tr>
<td>$s$</td>
<td>spin (angular momentum) qn</td>
<td>$s \in {0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots}$</td>
</tr>
<tr>
<td>$m, m_j$</td>
<td>secondary total angular momentum qn</td>
<td>$m_j = m_\ell + m_s$</td>
</tr>
<tr>
<td>$m, m_\ell$</td>
<td>secondary orbital (angular momentum) qn</td>
<td>$m_\ell \in {-\ell, -\ell + 1, -\ell + 2, \ldots, \ell - 2, \ell - 1, \ell}$</td>
</tr>
<tr>
<td>$m, m_s$</td>
<td>secondary spin (angular momentum) qn</td>
<td>$m_s \in {-s, -s + 1, -s + 2, \ldots, s - 2, s - 1, s}$</td>
</tr>
</tbody>
</table>

The hat notation $\hat{V}$ denotes an operator (rather than, say, a unit vector).

### 3.2 Mathematical Vocabulary

... 

### 3.3 Basics?

- **State**
- **Hamiltonian**
  The Hamiltonian doesn’t always represent the energy operator: it may include “interaction terms” that cannot strictly be called “potentials” (in the Newtonian sense where they are strictly position-dependent) - they may be velocity-dependent (in minimal coupling) or time-dependent (antihydrogen in a laser field), or...
  “Sometimes the Hamiltonian is allowed to be non-Hermitian to represent dissipative systems. When it corresponds to the energy operator of a system, the Hamiltonian is always Hermitian.” (from [http://minty.caltech.edu/Ph195/QDynamics2.pdf](http://minty.caltech.edu/Ph195/QDynamics2.pdf))
  HyperPhysics distinguishes between a “time-independent Hamiltonian” $\hat{H} = \hat{K} + \hat{P}$ and a “time-dependent Hamiltonian” $\dot{\hat{E}} = i\hbar \partial_t$. [http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/qmoper.html#c1](http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/qmoper.html#c1)
- **Operators** - Good development at [http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr2.html#c1](http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr2.html#c1)
- **Schrödinger Equation** - “Though the Schrödinger equation cannot be derived, it can be shown to be consistent with experiment.” ([http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr2.html#c1](http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/schr2.html#c1))
- **Hilbert Space** - a complete inner product space (countable basis implies separable, otherwise inseparable)
• **Physical Hilbert Space** - $V$ the space of functions (uncountably-infinite dimensional vectors) that can be normalized to either unity (proper vectors) or the Dirac delta function (improper vectors), over the field of complex numbers $\mathbb{C}$

• **Quantum Mechanical Linear Operators** - $T: V \rightarrow V$, $T \in \{\hat{H}, \hat{X}, \hat{K}, \hat{a}^\dagger, \hat{a}\}$

### 3.4 Dirac Notation and Terminology

- Bra
- Ket
- Properties (Associativity)
  - legal expressions
  - illegal expressions
  - semi-illegal expressions
- C-Number - “classical” or “commuting” number
- Q-Number - “quantum” or “queer” “number” - a mathematical object that does not commute in general (such as a matrix or an operator)

### 3.5 Continuing

- Hermitian Conjugate - (aka Hermitian Adjoint) $Q^\dagger = (Q^T)^*$ for a matrix operator
  Just as $\langle cV| = |V\rangle c^*$, $\langle \Omega V| = |V\rangle \Omega^\dagger$
  Let’s say $|\alpha\rangle$ is an eigenvector of the operator $\hat{A}$ with eigenvalue $a$, so $\hat{A}|\alpha\rangle = a|\alpha\rangle$. Then
  $\langle \alpha|\hat{A}^\dagger|\beta\rangle = \langle \hat{A}\alpha|\beta\rangle = \langle \beta|\hat{A}\alpha\rangle^* = (\langle \beta|\hat{A}|\alpha\rangle)^* = (a\langle \beta|\alpha\rangle)^* = a^* \langle \alpha|\beta\rangle$
- Hermitian Operator - $Q = Q^\dagger$
- Orthogonal Operator - $Q^T = Q^{-1}$
- Unitary Operator - $Q^\dagger = Q^{-1}$
- Rotation Operator
  - We assume “active rotations” rather than “passive rotations”, so that the physical system is actually rotated (instantaneously? with respect to its original (inertial?) reference frame)
    (I’m not sure what implications this leads to, about how there must be some torque on the system to accelerate it and then decelerate it, that this should take some time, etc.)
    $|\alpha\rangle_R = D(R)|\alpha\rangle$
    The symbol $D$ stems from the German word *Drehung*, meaning *rotation*. (Sakurai, pg 156)
- Symmetry Transformation - a transformation on a quantum mechanical vector space that leaves the Hamiltonian unchanged; such a transformation implies a conservation law (From Abers pg 159)
3.6 From My Quicknotes

3.6.1 Physics Terms

- First-quantized quantum mechanics (nonrelativistic, relativistic)
- Second-quantized theory
  (where the EM field is quantized at every point in space)
  (later... spacetime becomes a parameter, as time in the first-quantized theory ?)
- Canonical quantization (symplectic structure... see Wikipedia. symplectic, contact geometry Frobenius's theorem, foliation)
- Particle, Quasiparticle
- Quark, Hadron, Meson, Nucleon, Proton, Neutron,
- Lepton, Electron, Photon(?), Positron, etc.
- Fermion, Boson,
- Exciton, Photon(?), Phonon
- Positronium,
- Mass, Effective mass
- Hamiltonian (interaction terms in a Hamiltonian)
- Quantum numbers
- State (state vector)
  - Stationary state, Quasistationary state (or resonance) (Ziman pg 131)
  - Coherent state
  - Asymptotic state
  - Metastable states (do decay by collisions or by what are (misleadingly) called forbidden transitions)
    (lasers use metastable states)
  - Pure State (one energy eigenstate?), Mixed State (combination of two or more energy eigenstates?)
    (pure, mixed ensemble)
  - Ground state (Lowest energy state), Excited state Zero state (Reference state), Vacuum state, Null state, Empty state ket (see http://en.wikipedia.org/wiki/Creation_operator)
  - Bound state (Quasi-bound state (see below)), Continuum state
  - Wave packet (normalizable wave packet) (“We pretend to construct nrmlzbl wv pckts” Abers pg 292) See below (after these terms) for wave-packet explanation.
  - Localized wave train (See Feynman volume 1, 48-8)
  - “Dressed” state (Teller pg 122)
  - Separable:
    \[ \Psi(r, t) = \psi(r) \tau(t) = R(r) Y(\theta, \phi) \tau(t) = R(r) \Theta(\theta) \Phi(\phi) \tau(t) \]
    \[ \Psi(r, t) = \psi(r) \tau(t) = \bar{R}(r, t) \bar{Y}(\theta, \phi, t) = \bar{R}(r, t) \bar{\Theta}(\theta, t) \bar{\Phi}(\phi, t) \]
  - Separable, with spin:
    \[ \Upsilon(r, m, t) = \psi(r) \chi(m) \tau(t) \]
    \[ \Upsilon(r, m, t) = \bar{\psi}(r, t) \bar{\chi}(m, t) \]
  - With spin:
    \[ \Upsilon(r, \text{spin}, t) = \langle \Upsilon_1(r, \text{spin}, t); \Upsilon_2(r, \text{spin}, t); \ldots; \Upsilon_?(r, \text{spin}, t) \rangle \]
- Symmetry (of states or quantities or ...)
(spatial) parity: being even or odd under spatial inversion, aka “parity transformation”
apparently, this is not how parity is defined in other dimensions, e.g. 2-D (see Wikipedia)
time parity): being even or odd under time reversal (time parity transformation)
charge parity): being even or odd under charge conjugation (charge parity transf)
• Tunneling
• Spin (spin as angular momentum, spin as magnetic moment)
  “classical spins” (Callen pg 440)
• Valence band, Conduction band
• Chemical potential (ideal electron gas, Abers pg 446)
• Degeneracy pressure (Abers pg 447)
• Inelastic (scattering)
• Quasi-bound state (Sakurai pg 419)
• Scattering length (Sakurai pg 414)
• Scattering amplitude (angular ...something... density?)
• Partial wave amplitude
• Interaction, Coupling (volume-coupling, ex: Callen pg 53: Three volume-coupled systems) (Electric coupling?)
• Magnetic coupling (spin ang mom - orbital ang mom coupling, LS-coupling, spin-orbit interaction)
• Landé (gyroscopic) factor
• Form factor (Sakurai pg 431, nuclear form factor pg 433)
• Geometrical phase factor (Sakurai pf 469)
• Adiabatic potential (Sakurai pg 474)
• Hannay’s angle (Sakurai pg 480)
• Decay rate
• Decay modes
• Channels
• Differential decay rate
• Lifetime (time to reach \(1/e = 0.368\) of initial value)
• Half-life
• Spontaneous emission (emission stimulated by vacuum fluctuations)
• Stimulated emission
• Absorption
• Magnetic resonance (nuclear magnetic resonance, nmr)
• Rabi cycle, Rabi frequency
• Fast coordinates, Slow (nuclear) coordinates
• Transition Theory (Transformation Theory?), Scattering, Radiation, Collision, Etc Theories
• Transition, transition amplitude
• Forbidden transition, (Even more forbidden transition)
• T-matrix, Transition matrix (“the T-matrix satisfies certain algebraic relations, which provide valuable formal connections b/t elementary partial-wave scattering theory, Green functions, and the general theory of the S-matrix”)
(The evolution of quantum states over very long times - in particular, scattering and decays of long-lived
excited states - requires methods more powerful than those of potential scattering or the semiclassical
treatment. Abers pg 291)
\[ H = H^0 + H^1 \] (it's not important that \( H^1 \) be small, it's important that it have a small range. 9.2.1)
(Abers section 9.2.1 introduce a Step function for early times and a Regulator for late times)

- S-matrix, Scattering matrix
- K-matrix, or Reaction matrix (Ziman pg 130) (physically represents the total effect of the scattering
potential when at the heart of a standing wave system - i.e., with both incoming and outgoing spherical
waves as described by the propagator 4.133, “non-causal Green function”)
- Berry’s phase
- Degenerate (degeneracy) (also a mathematical term)
- Fine Structure, Splitting, Emission
  (spin-orbit coupling plus relativistic energy correction)?
  (Darwin term, contact term)
  (depends on \( n \) and \( j \) but, surprisingly, not on \( \ell \))
- Hyperfine Structure, Split, Emit (spin-spin coupling of nucleus and electron)
- Singlet, Doublet, etc. Multiplets (parahelium, orthohelium, etc.) (spectral lines/quantum states)
- Stark shift
- Stark broadening
- Lamb shift (can be computed in relativistic QFT, Abers pg 217) (Lamb-Rutherford...)
- Regularized energy (Abers pg 399)
- Reduced matrix element (Wigner-Eckart theorem, math term?)

Such a system of waves forms a crest which propagates itself with quite a different velocity from that
of its component waves, this velocity being the so-called group velocity. Such a wave crest represents a
material point which is thus either formed by it or connected with it, and is called a wave packet. De
Broglie now found that the velocity of the material point was in fact the group velocity of the matter-wave.

3.6.2 Mathematical terms

- Vector space
- Hilbert space (plus continuum-normalized states)
  - normalizable (continuum normalization, ...)
- Fock space (a space with states containing arbitrary numbers of particles connected by creation and
  annihilation operators. Abers pg 439)
- Hermitean conjugate
- Hermitean
- Degenerate
- Diagonalizable
- Diagonal
- Reduced matrix element (Wigner-Eckart theorem, physics term?)
- Secular determinant (Ziman pg 121)
- Resolvent operator/matrix (Ziman pg 122)
- Green function
- Non-causal Green function (Ziman pg 130, 4.133)
• Perturbation Theory

3.6.3 Equations, Formulas, or Rules

• Schrödinger’s Equation
• Blackbodies, Wein’s Law, Stefan’s Law?
• Fermi’s golden rule (golden rule number two) (transition rate, in first Born approximation)
• golden rule number one (transition rate, in second Born approximation)?
• Wigner-Eckart selection rules
• Hund’s rule
• Transition rules
• Dirac Fine-Structure Formula
• Thomas-Reiche-Kuhn sum rule (Sakurai pg 338)
• Lippman-Schwinger equation
• Dyson equation (Ziman 3.128, 4.136)
• Faxén-Holtsmark formula (Ziman pg 127)

3.6.4 Theorems

• Bell’s Theorem
• No-level crossing theorem (time-independent perturbation theory)
• \( H' \geq E_0 \) (variational methods) (Sakurai pg 313)
• Adiabatic Theorem

3.6.5 Principles

• Heisenberg uncertainty (indeterminacy, variance) principle (generalized)
• Correspondence principle (in the (statistical/large number/thermodynamic)? limit, QM → CM)
• Consistency condition (tot ang mom commutation rels must be same as that of orbital ang mom)
• Pauli exclusion principle

3.6.6 Approximations

• Electric dipole approximation (Sakurai pg 338)
• The Born Approximation
  (Born series, far-field solution of . . . assuming localized potential)
  (Spherically symmetric potential?)
  (First Born approximation, Second . . . )
• Born-Oppenheimer Approximation (Sakurai pg 474)
• Adiabatic approximation (Sakurai pg 473)
• Dyson Series (time-dep pert theory) (Sakurai pg 325) (approx?)
• Perturbation expansion
• Brillouin-Wigner perturbation expansion (Ziman pg 55)
3.6.7 Effects

- Photoelectric effect
- Paramagnetic resonance (Gasiorowicz pg 246)
- (quadratic) Stark effect
  (electric field - electric dipole mom coupling)
  (nondegenerate time-independent perturbation theory)
- (linear) Stark effect
  (degenerate time-independent perturbation theory)
  (Landé’s interval rule)? (Sakurai pg 306)
- (Anomalous) Zeeman effect
  (linear, quadratic)? (magnetic field - orbit coupling)
  (magnetic field - spin coupling)?
- Fine structure
  (Paschen-Back effect, limit)
  (van der Waals’ interaction)
- Magnetic anomaly of the spin (Landé g-factor) (Matthews pg 84)
- Hyperfine structure (splitting)
- Thomas precession
- Nuclear magnetic resonance, spin magnetic resonance (resonance condition Sakurai pg 321)
- Ramsauer-Townsend effect (Sakurai pg 413)
- Lamb shift
- (Dynamical) Jahn-Teller effect (Sakurai pg 475)
- Aharonov-Bohm effect
- Mössbauer effect (resonant, recoil-free emit absorb of gamma rays by atom bound in solid form)
- Quantum Zeno effect
  (Sakurai pg 311)

<table>
<thead>
<tr>
<th>Dominant Interaction</th>
<th>Almost good</th>
<th>No good</th>
<th>Always good</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weak $B$</td>
<td>$H_{LS}$</td>
<td>$J^2$ (or $\mathbf{L} \cdot \mathbf{S}$)</td>
<td>$L_z$, $S_z^*$</td>
</tr>
<tr>
<td>Strong $B$</td>
<td>$H_B$</td>
<td>$L_z$, $S_z$</td>
<td>$J^2$ (or $\mathbf{L} \cdot \mathbf{S}$)</td>
</tr>
</tbody>
</table>

*The exception is the stretched configuration, for example, $p_{3/2}$ with $m = \pm 3/2$. Here $L_z$ and $S_z$ are both good; this is because magnetic quantum number $J_z$, $m = m_l + m_s$ can be satisfied in only one way.
(Sakurai pg 319)

<table>
<thead>
<tr>
<th>State ket</th>
<th>Heisenberg picture</th>
<th>Interaction picture</th>
<th>Schrödinger picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>No change</td>
<td>Evolution determined by $V_I$</td>
<td>Evolution determined by $H_0$</td>
<td>Evolution determined by $H$</td>
</tr>
<tr>
<td>Evolution determined by $H$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.6.8 Questions

- What is spin?
- What is a singlet, doublet, etc? (parahelium, orthohelium, Sakurai pg 369)
- Why?... “Since $H$ is Hermitean, it has a complete set of eigenstates $|k\rangle$ such that $H|k\rangle = E_k |k\rangle$ and $I = \sum_k |k\rangle \langle k|$” (Abers pg 293)
• page 16 of notebook: especially, Why are eigenstates of $H$ orthogonal (orthonormal)?

4 Theoretical Summary

5 Early Quantum Theory

• Planck’s quantization was seen by him as a math trick, and a physical analysis seemed out of reach.
• Einstein “quantized” electromagnetism
• Bohr-Sommerfeld Model and the Sommerfeld-Wilson Quantization Condition
  – Elliptical quantized orbits
  – $\oint p \, dq = nh$
  – “up to first-order perturbation, the Bohr-Sommerfeld model and quantum mechanics make the same predictions for the spectral line splitting in the Stark effect. At higher-order perturbations, however, the Bohr-Sommerfeld model and quantum mechanics differ, and measurements of the Stark effect under high field strengths helped confirm the correctness of quantum mechanics over the Bohr model.
  – The Bohr-Sommerfeld quantization condition as first formulated can be viewed as a rough early draft of the more sophisticated condition that the symplectic form of a classical phase space $M$ be integral; that is, that it lies in the image of $\hat{H}^2(M, \mathbb{Z}) \to \hat{H}^2(M, \mathbb{R}) \to H^2_{DR}(M, \mathbb{R})$, where the first map is the homomorphism of Čech cohomology groups induced by the inclusion of the integers in the reals, and the second map is the natural isomorphism between the Čech cohomology and the de Rham cohomology groups. This condition guarantees that the symplectic form arise as the curvature form of a connection of a Hermitian line bundle. This line bundle is then called a prequantization in the theory of geometric quantization.”

• Heisenberg’s Quantum Mechanics (Matrix Mechanics)
• Schrödinger’s Quantum Mechanics (Wave Mechanics)

6 Transformation(?) from Classical to Quantum Mechanics

6.1 Postulates

• Physical systems are made of particles (molecules, atoms, elementary particles) (should I start with particle physics here?)
  From Shankar Chapter 4
### Postulates of Nonrelativistic (One-Particle) Theories

<table>
<thead>
<tr>
<th>Classical Mechanics</th>
<th>Quantum Mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. The state of a particle at any given time is specified by the two variables ( x(t) ) and ( p(t) ), i.e., as a point in a 2D phase space.</td>
<td>I. The state of a particle is represented by a vector (</td>
</tr>
</tbody>
</table>
| II. Every dynamical variable \( \omega \) is a function of \( x \) and \( p \): \( \omega = \omega(x, p) \). | II. The independent variables \( x \) and \( p \) of classical mechanics are represented by Hermitian operators \( X \) and \( P \) with the following matrix elements in the eigenbasis of \( X \):
\[
\langle x | X | x' \rangle = x \delta(x - x')
\]
\[
\langle p | P | p' \rangle = -i \hbar \delta(x - x')
\]
The operators corresponding to dependent variables \( \omega(x, p) \) are given Hermitian operators
\[
\Omega(X, P) = \omega(x \rightarrow X, p \rightarrow P)
\]
| III. If the particle is in a state given by \( x \) and \( p \), the (ideal classical) measurement of the variable \( \omega \) will yield a value \( \omega(x, p) \). The state will remain unaffected. | III. If the particle is in a state \( |\psi\rangle \), (ideal quantum) measurement of the variable (corresponding to) \( \Omega \) will yield one of the eigenvalues \( \omega \) with probability \( P(\omega) \propto |\langle \omega | \psi \rangle|^2 \). The state of the system will change from \( |\psi\rangle \) to \( |\omega\rangle \) as a result of the measurement. |
| IV. The state variables change with time according to Hamilton’s equations:
\[
\dot{x} = \frac{\partial \mathcal{H}}{\partial p}
\]
\[
\dot{p} = -\frac{\partial \mathcal{H}}{\partial x}
\] | IV. The state vector \( |\psi(t)\rangle \) obeys the Schrödinger equation
\[
i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \mathcal{H} |\psi(t)\rangle,
\]
where \( \mathcal{H}(X, P) = \mathcal{H}(x \rightarrow X, p \rightarrow P) \) is the quantum Hamiltonian operator and \( \mathcal{H} \) is the Hamiltonian for the corresponding classical problem. |

Note:
- State vectors contain information that yields the probability of every possible result for every possible measurement/experiment
  - A single ket \( |\psi\rangle \) represents the state of the particle in Hilbert space, and it contains a statistical prediction for all observables. To extract this information for any observable, we must determine the eigenbasis of the corresponding operator and find the projection of \( |\psi\rangle \) along all its eigenkets.
- Proper, improper vectors/states
- Principle of superposition
- Only possible values of \( \omega \) are the eigenvalues of \( \Omega \)
- Complication 1: The recipe \( \Omega(x \rightarrow X, p \rightarrow P) \) is ambiguous.
- Complication 2: The operator \( \Omega \) may be degenerate.
• Complication 3: The eigenvalue spectrum of $\Omega$ may be continuous.
• Complication 4: The quantum variable $\Omega$ may have no classical counterpart.

6.2 Quantization Methods
1. Canonical quantization ("Dirac method"?) (→ Weyl Quantization)
2. Covariant canonical quantization
3. Path integral quantization
4. Geometric quantization
5. Schwinger’s variational approach
6. Deformation Quantization (Weyl Quantization?)
7. Quantum statistical mechanics approach

• Canonical → Preserves symplectic structure
• Poisson brackets → Commutators (with $i\hbar$’s)
• Quantities → Operators
• Liouville’s Theorem → Ehrenfest Theorem

6.3 Interpretations
• Copenhagen
  – Correspondence Principle
• ...

6.4 Other Ideas
• Quantum logic, Quasi-set theory

7 Representations, Expansions, and Approximations

7.1 Schrödinger, Heisenberg, and Interaction (Dirac) Pictures
Schrödinger equation becomes the Schwinger-Tomonaga equation in the interaction picture.

Schrödinger Picture
\[
\begin{align*}
|\Psi(t)\rangle_S &= e^{-iHt/\hbar} |\Psi(0)\rangle_S \\
\langle\Psi(t)|_S &= e^{iHt/\hbar} \langle\Psi(0)|_S
\end{align*}
\]

or
\[
\begin{align*}
|\Psi(0)\rangle_S &= e^{iHt/\hbar} |\Psi(t)\rangle_S \\
\langle\Psi(0)|_S &= e^{-iHt/\hbar} \langle\Psi(t)|_S
\end{align*}
\]
• Must it be $\langle\Psi(t)|_S = \langle\Psi(0)|_S e^{iHt/\hbar}$ and similarly elsewhere???
Heisenberg Picture

State vectors $|\Psi\rangle$, given that $H$ is time-independent:

$|\Psi(t)\rangle_S = e^{-iHt/\hbar} |\Psi\rangle_H$ or $|\Psi\rangle_H = e^{iHt/\hbar} |\Psi(t)\rangle_S$

$\langle\Psi(t)|_S = e^{iHt/\hbar} \langle\Psi|_H$ or $\langle\Psi|_H = e^{-iHt/\hbar} \langle\Psi(t)|_S$

Coordinate and momentum states:

$|\phi(t)\rangle = e^{iHt/\hbar} |\phi\rangle$ or $|\phi\rangle = e^{-iHt/\hbar} |\phi(t)\rangle$

$\langle\phi(t)| = \langle\phi| e^{-iHt/\hbar}$ or $\langle\phi| = e^{iHt/\hbar} \langle\phi(t)|$

• Or should it be $|\phi, t\rangle$???

Operators $\hat{\phi}$ and $\hat{\pi}$:

$\hat{\phi}(x) = \hat{\phi}(x, t) = e^{iHt/\hbar} \hat{\phi}(x) e^{-iHt/\hbar}$

$\hat{\pi}(x) = \hat{\pi}(x, t) = e^{iHt/\hbar} \hat{\pi}(x) e^{-iHt/\hbar}$

Thus

$\langle\phi| \Phi(t)\rangle_S = \langle\phi| e^{-iHt/\hbar} |\Psi\rangle_H = \langle\phi(t)| e^{iHt/\hbar} e^{-iHt/\hbar} |\Psi\rangle_H = \langle\phi(t)| \Psi\rangle_H$

$\langle\phi| \hat{\pi}(x) \Phi(t)\rangle_S = \langle\phi(t)| e^{iHt/\hbar} \hat{\pi}(x) e^{-iHt/\hbar} |\Phi\rangle_H = \langle\phi(t)| \hat{\pi}(x, t) |\Phi\rangle_H$

Heisenberg Equation of Motion:

$i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]$

Interaction Picture

7.2 Special Functions, Et Cetera

The spherical harmonics $Y^l_m(\hat{n})$ are the spherical components of the unit vector $\hat{n}$.

8 Angular Momenta

9 Spin

• Ryder [?] pg 40: “An analysis of [the Poincaré] group gives a true understanding of, and some surprising insights into, the nature of spin.”

• Thought: If spin is not due to current, then there should not be any torque when a spin magnetic moment is in a magnetic field.

• Is spin due to the decay of particles into charged particles that spin around before they recombine? (Where would that angular momentum go when they recombine?)

Spin is short for spin angular momentum (and, maybe, spin magnetic moment). Spin relates to angular momentum ($\mathbf{J}_{\text{spin}} = \mathbf{S}$) and magnetic dipole moment ($\mathbf{\mu}_{\text{spin}} = -g_p \mu_p \mathbf{S}$). The relationship between spin angular momentum and its associated magnetic dipole moment is in some ways classical (their relative directions can be explained by the classical idea of rotation) and in other ways not classical (their relative magnitudes cannot be explained with this analogy, and are explained with... the Dirac equation and QFT?, not Thomas precession).

(Why do we force the magnetic dipole moment vector operator to be parallel to the spin vector operator?)
The magnetic dipole moment of a particle $p$ with spin $S$ is

$$\mu = \mu_{\text{orb}} + \mu_{\text{spin}} = -\mu_p \mathbf{L} - g_p \mu_p S$$

where $g_p$ is the g-factor for particle $p$ and $\mu_p$ may be the the Bohr magneton $\mu_B$ or the nuclear magneton $\mu_N$:

$$\mu_B = \frac{e\hbar}{2m_e}$$

$$\mu_N = \frac{e\hbar}{2M}$$

where $M$ is the mass of the nucleon in question.

The gyromagnetic ratio (or magnetomechanical or magnetogyric ratio) expresses the ratio of an object’s magnetic dipole moment to its angular momentum.

<table>
<thead>
<tr>
<th>particle</th>
<th>g-factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>e electron</td>
<td>$g_e \approx 2 \times (1 + \frac{\alpha}{2\pi}) \approx 2.0023$</td>
</tr>
<tr>
<td>p proton</td>
<td>$g_p \approx 2 \times 2.8 = 5.6$</td>
</tr>
<tr>
<td>n neutron</td>
<td>$g_n \approx -2 \times 1.91 = -3.82$</td>
</tr>
<tr>
<td>N nucleus</td>
<td>$g_N$: many different values</td>
</tr>
</tbody>
</table>

$$g_e = \frac{2}{2} \left[ 1 + \frac{\alpha}{2\pi} - (0.328478445) \left( \frac{\alpha}{\pi} \right)^2 + (1.18311) \left( \frac{\alpha}{\pi} \right)^3 + \cdots \right]$$

$S = \hbar \frac{\sigma}{2}$

9.2 Spin Phenomena

- **Stern-Gerlach experiment**: canonical quantum mechanics experiment, having to do with spin.
- **Anomalous Zeeman Effect** (and its inverse)
- **Larmor precession**: the precession of the magnetic moments of electrons, atomic nuclei, and atoms around the direction of an external magnetic field.

$$\tau = \mathbf{\mu} \times \mathbf{B}$$

$$\mathbf{B} = B_0 \hat{z} \Rightarrow \omega_0 = \gamma B_0, \gamma = g \frac{\mu_B}{h}$$
– FMR (ferromagnetic resonance): a spectroscopic technique to probe the magnetization of ferromagnetic materials. It is a standard tool for probing spin waves and spin dynamics. FMR is very similar to nuclear magnetic resonance except FMR probes the magnetic moment of electrons and NMR probes the magnetic moment of the proton.
– NMR (nuclear magnetic resonance)

• **Rabi oscillation**: the cyclic behaviour of a two-state quantum system in the presence of an oscillatory driving field. (Two-state wrt energy? i.e., non-energy-degenerate?)
– When an atom (or some other two-level system) is illuminated by a coherent beam of photons, it will cyclically absorb photons and re-emit them by stimulated emission. One such cycle is called a Rabi cycle and the inverse of its duration the Rabi frequency of the photon beam.
– This mechanism is fundamental to quantum optics. It can be modelled using the Jaynes-Cummings model and the Bloch vector formalism.
– The effect is important in quantum optics, magnetic resonance imaging and quantum computing.
– Norman F. Ramsey, modified the Rabi apparatus to increase the interaction time with the field. The extreme sensitivity due to frequency of the radiation makes this very useful for keeping accurate time, and is still used today in atomic clocks.
– On Wikipedia’s “Rabi Cycle” article, see also: Larmor frequency, Laser pumping, Optical pumping, Rabi problem, Bloch sphere, Atomic coherence

• **Spin-Statistics Connection** (Theorem)

10 **Transition Theory: See HydrogenTrans Lecture**

Non-Degenerate Perturbation Theory

\[ \Delta E_n^{(1)} = \langle \psi_n^0 | H_1^1 | \psi_n^0 \rangle \]
\[ \Delta E_n^{(2)} = - \sum_{m \neq n} \frac{|\langle \psi_m^0 | H_1^1 | \psi_n^0 \rangle|^2}{E_m^0 - E_n^0} \]

Wigner-Eckart theorem:

\[ \langle \alpha', j', m' | T_k | \alpha, j, m \rangle = \frac{1}{\sqrt{2j'+1}} \langle \phi_{j', m'}^{\alpha', j'} | \langle \alpha', j' | T_k | \alpha, j \rangle \]

Wigner-Eckart selection rules:

\[ \langle \alpha', j', m' | T_k^q | \alpha, j, m \rangle = 0 \text{ unless } \begin{cases} 1. & m' = m + q, \text{ and} \\ 2. & |j - k| \leq j' \leq j + k. \end{cases} \]
\[ \langle \alpha', j', m' | T_k^q | \alpha, j, m \rangle = 0 \text{ unless } \begin{cases} 1. & \Delta m = q, \text{ and} \\ 2. & |\Delta j| \leq k \leq \sum j. \end{cases} \]

\[ \Delta m \equiv m' - m, \Delta j \equiv j' - j, \sum j = j + j'. \] Condition 2 can be stated loosely as “\(j', j, \text{ and } k\) can form a triangle”. Now, interpret these rules physically.

Using time-dependent perturbation theory... we have \( H = H_0^1 + H_1 \) (where \( H_1 \) is localized but not necessarily small), and eigenstates \( |\phi_a\rangle \) and \( |\phi_b\rangle \) of the “unperturbed” Hamiltonian \( H_0^1 \). The system starts at time \( t_0 \) in state \( |\phi_a\rangle \) and we’d like to know the probability that the system is in state \( |\phi_b\rangle \) at some time \( t \) long after the perturbing interaction has taken place...
The transition amplitude $a_{ba}(t)$ that the system, initially in the state $|φ_a⟩$, be in the state $|φ_b⟩$ at time $t$ is

$$a_{ba}(t) = \langle φ_b | U(t - t_0) | φ_a⟩ = \langle φ_b | e^{-iH(t-t_0)} | φ_a⟩$$

$$\approx \frac{i}{2π} \int_{-∞}^{∞} e^{-iωt} A_{ba}(ω)dω$$

$$= \lim_{ε→0} \frac{i}{2π} \int_{-∞}^{∞} e^{-iωt} \left[ e^{iωt_0} \left| φ_b \right⟩ \left( \frac{1}{ω - H + iε} \left| φ_a \right⟩ \right) \right] dω$$

$$= \lim_{ε→0} \frac{i}{2π} \int_{-∞}^{∞} e^{-iω(t-t_0)} G(ω)_{ba} dω$$

$$\approx \text{more stuff}$$

Transition matrix:

$$T(ω) ≡ H′G(ω)G^0(ω)^{-1}$$

$$= H′ + H′G(ω)H′$$

$$= H′ + H′G^0(ω)T(ω)$$

(which expressions are useful?)

Born Series:

$$T(ω) = H′ + \cdots$$

First approximation uses first term, second approximation uses first two terms, and so on.

The Green operator

$$G(ω) = G^0(ω) + G^0(ω)T(ω)G^0(ω)$$

Lippmann-Schwinger equation:

$$|ψ_a⟩ = |φ_a⟩ + G^0(ω_a)H′ |ψ_a⟩$$

The transition rate in terms of the transition matrix:

$$Γ = \frac{d}{dt} P_{ba} = 2 |T_{ba}(ω_a)|^2 \frac{sin[(ω_b - ω_a)(t - t_0)]}{ω_b - ω_a} \left| T_{ba}(ω_a) \right|^2 δ(ω_b - ω_a)$$

Fermi’s Golden Rule (golden rule number two):

$$Γ = 2π \left| H′_{ba} \right| δ(ω_b - ω_a)$$

$$Γ_{ba} = 2π \int \left| H′_{ba} \right|^2 δ(ω_b - ω_a)$$

(transition rate, in first Born approximation)

(Abers 11.90, formula for single photon emission rate, using spinless nonrelativistic description of the electron, and first Born approximation (Fermi’s golden rule))

$$Γ = \frac{e^2ω}{2πm^2} \sum_α \int \left| \left( ψ_f | e^{-i\mathbf{k} \cdot \mathbf{r}} \hat{e}^*_α(\mathbf{k}) | ψ_i \right) \right|^2 dΩ$$

$$= \frac{e^2ω}{2πm^2} \sum_α \int \hat{e}^*_α(\mathbf{k}) \cdot \left( ψ_f | pe^{-i\mathbf{k} \cdot \mathbf{r}} | ψ_i \right) \left| ψ_i \right|^2 dΩ$$

18
(apparently the \( p \)'s and \( r \)'s from the exponent can commute since, as explained on Abers pg 370, \( A \) and \( p \) commute due to the gauge condition) (\( k = \omega \mathbf{n} \))

Furthermore, using dipole approximation, with \([r, H_0] = ip/m\),

\[
\Gamma = \frac{e^2 \omega^3}{2\pi} \frac{8\pi}{3} |r_{fi}|^2 = \frac{4\omega^3}{3} |r_{fi}|^2
\]

(What does this “dipole approximation” mean physically?)

10.1 (Bound State?) Perturbation Theory

10.2 Scattering Theory

Scattering matrix:

10.2.1 Potential Scattering, Born Series

10.3 Collision Theory (subtopic of scattering?)

10.4 Decays of Excited States

10.4.1 Radioactive Decay

11 Equations, Laws, and Formulas

- **Ehrenfest Theorem**

\[
d_t \langle \hat{A} \rangle = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle + \langle \partial_t \hat{A} \rangle
\]

where \( \hat{A} \) is any physical quantum operator and \( \hat{H} \) is the Hamiltonian.

This is easily derivable starting with full expression for \( \langle A \rangle \) and using the Schrödinger equation.

Use the generic Hamiltonian \( \hat{H} = p^2/2m + V(x, t) \) and, after a little work, get a form of Newton’s second law:

\[
d_t \langle p \rangle = \langle -\nabla V(x, t) \rangle = \langle F \rangle
\]

illustrating the correspondence principle.

- **Lennard-Jones Potential**

\[
u(r) = 4u_0 \left[ \left( \frac{r_0}{r} \right)^{12} - 2 \left( \frac{r_0}{r} \right)^6 \right]
\]

where \( u_0 \) is the well depth and \( r_0 \) is the

12 Applications and Phenomena

- Lasers (i.e., LASERs)

13 Advanced Topics

13.1 Phenomenology

- **Master Eqns**: (in physics in general) a phenomenological first-order differential equation describing the time-evolution of the probability of a system to occupy each one of a discrete set of states.
One generalization of the master equation is the **Fokker-Planck eqn** which describes the time evolution of a continuous probability distribution.

- **Lindblad Eqn**: a master equation for the time evolution of a mixed state in quantum mechanics. (Related to Liouville Thm and Ehrenfest Thm.)
  - Rather, the Lindblad form is the most general form that a master equation is allowed to take in quantum mechanics to describe non-unitary (dissipative) evolution of the density matrix \( \rho \) (such as ensuring normalization and hermiticity of \( \rho \)).
    
    \[
    \frac{d}{dt} \rho = -\frac{1}{i\hbar} [\rho, H] - \frac{1}{\hbar} \sum_{n,m} h_{n,m} (\rho L_m L_n + L_n L_m \rho - 2L_n \rho L_m) + \text{h.c.}
    \]

    where \( L_m \) are operators and \( h_{m,n} \) are constants that chosen to make the model in question accurate, and h.c. denotes the Hermitian conjugate of the previous terms in the summation (correct?).

  - The most common Lindblad equation is that describing the damping of a quantum harmonic oscillator, it has \( L_0 = a, L_1 = a^\dagger \), \( h_{0,1} = -(\gamma/2)(\bar{n} + 1) \), \( h_{1,0} = -(\gamma/2)\bar{n} \) with all others \( h_{n,m} = 0 \). Here \( \bar{n} \) is the mean number of excitations in the reservoir damping the oscillator and \( \gamma \) is the decay rate. Additional Lindblad operators can be included to model various forms of dephasing and vibrational relaxation. These methods have been incorporated into grid-based density matrix propagation methods.

### 14 Problem Solving

- Systems of spins: Normally, it’s easiest to rewrite dot products of different spins as squares of the various spins. Examples: ; And a simple counter-example: .

### References

