# Notes on Classical Mechanics

Newtonian, Lagrangian, and Hamiltonian Mechanics, and Classical Field Theory

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1 Ideas and Questions

1.1 Questions

- Now that I know what $dx$ and $df$ are (one-form fields), what is $\delta f$ in functional calculus? (WRT the functional derivative, etc.) How do the limits of integration in the calculus of variations come into play?

- Explain the advantages (disadvantages?) of the Hamiltonian and Lagrangian formalisms over the Newtonian formalism (solving certain problems, gaining information from Lagrangian/Hamiltonian that is not otherwise available?)

- Why not a generalized potential $U = U(q, \dot{q}, \ddot{q}, \ldots, t)$?

- The Work-Kinetic Energy Principle can be generalized (Ch 1, prob 4). Is this a generally useful result? If so, what should we call it? How should we think about it?

- In what circumstances, and to what extent, are variables such as $q$ and $\dot{q}$ considered independent?

- Examine the Lagrangians of QFT (and any other Lagrangians I find) to see if they satisfy the condition for transformation into the Hamiltonian formalism. (Lagrangian is concave up or concave down wrt velocities, i.e., second derivative wrt velocities is either always positive or always negative.)

- How do we come up with $U$ for the charged particle in an e-m field?

- Why is the [Hamiltonian for a nonrelativistic charged particle in electromagnetic fields] not $T + U$ but is the total energy?

- When is the Hamiltonian
  - equal to the total energy of the system in question? and when is it not?
  - equal to the total mechanical energy of the system in question? and when is it not?
  - conserved? and when is it not?

  * A case: If the Lagrangian does not explicitly depend on time AND if the generalized coordinates appearing in the Lagrangian are unconstrained, then the Hamiltonian is conserved.
If you include the constrained angular coordinate in the Lagrangian as a generalized coordinate, then the theorem is no longer true. You have to use Lagrange multipliers or some other technique to deal with the constrained system i.e. to find equations of motion.


- Are scientists trying to come up with “the most fundamental Hamiltonian” (of the universe), and should it be conserved? (Perhaps once they come up with this fundamental, conserved Hamiltonian, then they'll know how to break it up and to examine subsystems (of the universe), which may have nonconserved Hamiltonians.)
- What is the difference between a dissipation function and a generalized potential?
- Is D’Alembert’s principle equivalent to Newton’s second law, as Wikipedia claims. (Article: D’Alembert’s principle)

1.2  Things to Look Into

From Goldstein’s [1] index:
- (Principle of) Virtual work
- Virial of Claussius
- Symplectic notation, for Hamilton’s equations

Show principle of maximal aging implies principle of least action. (http://www.eftaylor.com/index.html)

1.3  Unit Questions

Are they really the same? If not, how are they different?
- torque ($\tau = r \times F$) has the “same” units as energy ($W = \int F \cdot ds$): $m \times N "=\cdot N \cdot m$
- angular momentum ($L = r \times p$) has the “same” units as action ($S = \int L \, dt$): $m \times kg \cdot \frac{m}{s} "=\cdot J \cdot s$
- volumetric flux density (... meters cubed of substance per second per meter squared of surface through which the substance passes) has the “same” units as velocity (meters traversed per second): $\frac{m^3}{s^2} / m^2 "=\cdot m/s$

2  The Big Picture

Classical mechanics includes the general theory of relativity and what else? (See notes on relativity for details of relativistic mechanics.)

2.1  Historical Development

D’Alembert’s principle? Hamilton-Jacobi eqns?
- French mathematician Adrien-Marie Legendre (1752-1833)
- German mathematician Carl Gustav Jacob Jacobi (1804-1851)
- 1833 - The Irish mathematician William Rowan Hamilton (1805-1865) invents a reformulation of classical mechanics – Hamiltonian mechanics.
- 1834 - Hamilton’s principle
• 1867 - The terms **generalized coordinates**, **generalized velocities**, and **generalized momenta** were introduced by Sir William Thomson (later, Lord Kelvin) and P. G. Tait in their famous treatise *Natural Philosophy*.

• 1870 - Rudolf Julius Emanuel Clausius (1822-1888) proves the scalar virial theorem.

3 Notation

- A set of generalized coordinates \( \{q_1, q_2, q_3, \ldots, q_n\} \) may be denoted \( \{q_j\}_{j \in \mathbb{N}_n} \), or \( q_j \), or \( q \), or simply \( q \). This last notation will be most common, and it should be clear from the context whether \( q \) represents one coordinate or a set of coordinates. (If unsure, assume \( q \) represents a set of coordinates rather than one coordinate.)
  - The same goes for canonical momenta: \( \{p_1, p_2, p_3, \ldots, p_n\} \) may be denoted \( \{p_j\}_{j \in \mathbb{N}_n} \), or \( p_j \), or \( p \), or simply \( p \).

- There is an ambiguity between the polar-angle component of Cartesian momentum and the canonical momentum conjugate to the polar angle, both of which may be denoted \( p_\theta \). (Similar ambiguities arise for other momenta.)
  - How shall I resolve this ambiguity? Use \( p_\theta \) or \( \dot{\theta} \)?

- Dot notation (Newton’s fluxion)
  \[
  \dot{q} \equiv \frac{dq}{dt}
  \]
  (I’m not sure, but this might also sometimes be used to denote the partial time derivative.)

What about when we go from a particle Lagrangian to a field Lagrangian density:

\[
\frac{\partial L}{\partial \dot{q}} \rightarrow \frac{\partial L}{\partial (\partial_t \phi)}
\]

- \( \dot{\phi} \equiv \partial_t \phi \)

- Total derivative
  \[
  \frac{d}{dt} \equiv \frac{d}{dt}
  \]
  Change over time given an arbitrary change in all other variables, if any.

- Partial derivative
  \[
  \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t}
  \]
  Change over time given that all other variables, if any, are constant.

4 Mathematics

- Standard Calculus
- Functional Calculus
- Differential Geometry
- (Lie Groups, Algebras, Representations?)
- Small Theorems
  - Euler’s thm - If \( f(y_k) \) is a homogeneous function of the \( y_k \) that is of degree \( n \), then \( \sum_k y_k \frac{\partial f}{\partial y_k} = nf \).
5 Terms and Quantities

- Position
- Time
- Velocity, Acceleration, Jerk, etc.
- Inertia
- Momentum
- Force
- Mass
- Moments (of force, of mass, of inertia, etc)
- System
- Degrees of Freedom
- Coordinates
- Generalized Coordinates
- Equation of Motion
- Constraint
- Equation of Constraint
- Holonomic Constraints and Systems
- Monogenic Forces and Systems
- $T(\mathbf{v}), T(\dot{\mathbf{q}})$ Kinetic Energy
  The energy of motion of objects relative to each other.
- $V(\mathbf{r}), V(q)$ Potential Energy (Conservative Potential)
- $U(q, \dot{q}, t)$ Generalized Potential
  (also $U(q, \dot{q})$ “Velocity-dependent Potential”)
- $\mathcal{F}$ Dissipation Function
  (also Rayleigh’s dissipation function, or viscous dissipation function)

- **Stress Tensor** - Energy-Momentum-Stress Tensor
  $T^{\mu\nu}$ is “the flux of 4-momentum $p^\mu$ across a surface of constant $x^\nu$.”
  Or $T^{\mu\nu}$ is “the flux of 4-momentum component $p^{\mu}$ across a surface of constant $x^{\nu}$.”
  $T^{00}$, the “flux of $p^0$ (energy) in the $x^0$ (time) direction”, is the rest-frame energy density
  Mechanical Stress Tensor - stress, strain, compression, pressure, viscosity (normal and tangential
  tractions, or equivalently, direct and shear stresses)
  See [http://www.bun.kyoto-u.ac.jp/~suchii/extrem.aging.html](http://www.bun.kyoto-u.ac.jp/~suchii/extrem.aging.html)
  Stress Holor $T^{ij}$ as a sub-holor of $T^{\mu\nu}$
  Dust and perfect fluid (which can be completely specified by two quantities, the rest-frame energy
  density $\rho$ and an isotropic rest-frame pressure $p$)
“Einstein and von Laue proposed that the problem might lie with the field equation, which, they suggested, should have the linear form $FT_{\text{matter}} = \rho$, where $F$ is some yet unknown function of $\phi$, and where $T_{\text{matter}}$ is the trace of the stress-energy tensor describing the density, momentum, and stress of any matter present.” (http://en.wikipedia.org/wiki/Nordström’s_theory_of_gravitation)

\[
T_{\mu\nu}^{\text{dust}} = p^{\mu} N^{\nu} = m n u^{\mu} u^{\nu} = \rho u^{\mu} u^{\nu}
\]

\[
T_{\mu\nu}^{\text{perfect fluid}} = (\rho + p) u^{\mu} u^{\nu} + p n^{\mu} n^{\nu}
\]

Scalar field theory stress tensor

\[
T_{\text{scalar}}^{\mu\nu} = \eta^{\mu\lambda} \eta^{\nu\sigma} \partial_{\lambda} \phi \partial_{\sigma} \phi - \eta^{\mu\nu} \left[ \frac{1}{2} \eta^{\lambda\sigma} \partial_{\lambda} \phi \partial_{\sigma} \phi + V(\phi) \right]
\]

EM stress tensor

\[
T_{\text{EM}}^{\mu\nu} = F^{\mu\lambda} F^{\nu\lambda} - \frac{1}{4} \eta^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}
\]

Energy density, energy flux (momentum density?)

Given Hilbert-Einstein Action

\[
S_H = \int \sqrt{-|g|} R \, d^4x
\]

\[
S = \frac{1}{16\pi G N} S_H + S_M
\]

where $S_M$ is the action for matter.

We have another definition of the stress tensor:

\[
T_{\mu\nu} = -2 \frac{1}{\sqrt{-|g|}} \frac{\delta S_M}{\delta g^{\mu\nu}}
\]

Spin tensor

Warning: In solid state physics and fluid mechanics, the stress tensor is defined to be the spatial components of the stress-energy tensor in the comoving frame of reference. In other words, the stress energy tensor in engineering differs from the stress energy tensor here by a momentum convective term.

6 Theoretical Summary

6.1 Abstract Mathematical View

- Newtonian... (vectors, affine space, tangent space)
- A Lagrangian is a time-dependent ?-form field on coordinate space(?).
- The Hamiltonian is a (time-dependent?) scalar on the cotangent bundle (of phase space?). The total space of a cotangent bundle naturally has the structure of a symplectic manifold. (Wikipedia: Differentiable Manifold)

6.2 Important Equations

- Generalized Coordinates

As opposed to numbering the particles

\[
r_1 = r_1(q_1, q_2, \ldots, q_{3N-k}, t)
\]
• Eqns of Constraint
  (What are the issues dealing with time dependence, etc?)
• Transformation Eqns (or Parametrization of Coordinates, a kind of Point Transformation)
  Given $N$ particles in 3D space and $k$ eqns of constraint

\[
\begin{align*}
  r_1 &= r_1(q_1, q_2, \ldots, q_{3N-k}, t) \\
  r_2 &= r_2(q_1, q_2, \ldots, q_{3N-k}, t) \\
  \vdots \\
  r_{3N-k} &= r_2(q_1, q_2, \ldots, q_{3N-k}, t)
\end{align*}
\]

where the independent coordinates

6.3 Newtonian Formalism
• Good for elucidating physical mechanisms (?)

6.4 Lagrangian Formalism
• Local, Differential Principles
  – Newton’s Laws (Principles of instantaneous forces and particulate motion(?)
  – D’Alembert’s principle
    * small virtual displacements about an instantaneous state (a point in configuration space)
      (and then integrate?)
• Global, Integral (or Variational) Principles
  – Hamilton’s principle
    * small virtual variations of the entire motion (through configuration space) of the system between
times $t_1$ and $t_2$ from the actual motion
      (This seems to force one to decide what the final and initial states should be.)

6.5 Hamiltonian Formalism
• Local, Differential Principles
  – ...
• Global, Integral (or Variational) Principles
  – Modified Hamilton’s principle

6.6 Advantages and Disadvantages of Variational Principle Formulation
From Soper [3], “…the principle of stationary action …does not apply to systems subject to frictional forces. To discuss such systems one must return to an $F = ma$ approach. Likewise we will have to go beyond Hamilton’s principle in Chapter 13 when we discuss field theories that include dissipative processes like viscosity, heat flow, and the flow of electric current through a resistor.”
• From Goldstein:
  – Question: How many of these “advantages” also apply to other formulations?
− **Coordinate Invariance**: Formulation refers to kinetic and potential energies, which are always definable and independent of coordinate system.

− **Universality of technique**: Lagrangian formulation can be extended easily to describe systems that are not normally considered in dynamics, such as elastic fields, the electromagnetic field, and field properties of elementary particles.

− **Formal similarity of phenomena**: Formal similarities between different kinds of systems (such as electrical circuits and mechanical systems) become apparent. (Terms normally reserved for electrical circuits, such as reactance and susceptance, are the accepted modes of expression in much of the theory of vibrations of mechanical systems.1)

− **Structural analogy of fields of study**: Lagrange’s and Hamilton’s principles together form a compact invariant way of implying the mechanical equations of motion. This works outside of mechanics also: variational principles can be used to express the “equations of motion,” whether they be Newton’s equations, Maxwell’s equations, or the Schrödinger equation. So, when a variational principle is used as the basis of formulation of all fields, they will exhibit, at least to some degree, a structural analogy.

− The methods of quantization were first developed for particle mechanics, starting essentially from the Lagrangian formulation of classical mechanics. By describing the electromagnetic field by a Lagrangian and corresponding Hamilton’s variational principle, it is possible to carry over the methods of particle quantization to construct a quantum electrodynamics.

• From Marion and Thorton:

− The Newtonian approach emphasizes an outside agency acting on a body (the force), while the Lagrangian method deals only with quantities associated with the body (the kinetic and potential energies). In certain situations it may not be possible to state explicitly all the forces acting on a body, as is sometimes the case for forces of constraint and as is normally the case for quantum mechanical systems where we normally know the energies but not the forces.

### 6.7 Other Stuff

From Abers, page 47, we have that the classical equation of motion is

\[ \dot{A}(t) = \{A, H\} + \partial_t \]

### 7 Newtonian Mechanics

#### 7.1 Forces, Newton’s Laws, and Conservation

• Newton’s Laws of Motion

− N1: Inertia
  
  * Statement of kinematics in absence of forces (interaction). Special case of N2.
  
  ⇒ CLM and CAM for elementary particles with no external forces. (Not observed, not interacting.)

− N2: Force (Action) and Momentum
  
  * How does one define a force and mass?

- N3: Coaction
  * WN3
  * SN3

- N2: Various forms of Newton’s Second Law (N2) depend on the Laws of Coaction (forms of Newton’s Third Law, N3)
  - Forms that are always valid: Single particle, Superposition of multiple particles (both “linear” and angular forms, correct?)

- WN3: Weak Law of Coaction
  - WN3: The forces that two particles exert on each other are equal and opposite.
  - ⇒ N2 for Systems of particles is valid (Using total mass, center of mass, net force)
  - In absence of external forces
    Conservation of total linear momentum assumes WN3 to be true
    (so the internal forces cancel)

- SN3: Strong Law of Coaction
  - SN3: The forces that two particles exert on each other are equal and opposite and directed along the line joining the particles.
  - ⇒ N2 for Angular momenta and forces (torque) is valid
  - In absence of external forces
    Conservation of total angular momentum assumes SN3 to be true
    (so the internal forces are central and cancel)

- N3 (in either form) DOES NOT hold for all forces (in the Newtonian picture) (…if you include everything such as momentum of fields, etc, then, in some sense, N3 always holds). (elaborate on this…)

- PLN2 ⇔ PAN2 ⇔ CPLM ⇔ CPAM
  PLN2 + WN3 ⇔ SLN2 ⇔ CTLM
  PAN2 + SN3 ⇔ SAN2 ⇔ CTAM

  C = conservation of, P = particulate, T = total, S = systemic/strong, W = weak, L = linear, A = angular, M = momentum, N = Newton’s, 2 = second law, 3 = third law

- Mechanical Energy:
  Note that in some circumstances, a force may be given by the gradient of a scalar function \( F = -\nabla V \) and the mechanical energy \( E = T + V \) may still be defined, but
  - \( W \neq \Delta V \)
  - \( E \) is NOT conserved

7.1.1 Examples
- \( F = -\nabla V \) where \( V = V(|\mathbf{r}_2 - \mathbf{r}_1|) = V(R) \) ⇒ SN3
- \( F = -\nabla V \) where \( V = V(|\mathbf{r}_2 - \mathbf{r}_1|, |\mathbf{v}_2 - \mathbf{v}_1|, |\mathbf{s}_2 - \mathbf{s}_1|) \) ⇒ WN3 only

7.2 Extraneous Material
- Center of mass \( \neq \) Center of gravity
8 Lagrangian Formalism

The Lagrangian formulation of classical mechanics

- (in its usual holonomic form) eliminates the constraining forces from the equations of motion
- utilizes scalar functions \((L, T, U)\), which can simplify problems
  (Describe how, why)

8.1 Fundamentals

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Geometric limits on the system where the forces enforcing these limits are not necessarily known ((apriori)) or something like that (always geometric?)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Coordinates</td>
<td>Coordinates of any type that fully distinguish a state of the system, consistent with the constraints</td>
</tr>
<tr>
<td>Virtual Displacement and Virtual Work</td>
<td>(\delta r, \delta q)</td>
</tr>
<tr>
<td>Variation and First Variation</td>
<td>(\delta F(q, t) = \frac{\partial F}{\partial q} \delta q + \frac{\partial^2 F}{\partial q^2} (\delta q)^2 + \cdots = \frac{\partial F}{\partial q} \delta q)</td>
</tr>
<tr>
<td>Functional Differentiation and Functional Integration</td>
<td></td>
</tr>
</tbody>
</table>

| Principle of Virtual Work       | Virtual-work-less constraining forces \(\sum_i \mathbf{F}_i^c \cdot \delta r = 0\), with equilibrium \(\mathbf{F} = 0\) |
|                                 | \(\Rightarrow \sum_i \mathbf{F}_i^o \cdot \delta r = 0\)                                                        |
|                                 | (IS this a sum over particles? If not, then the transformation to general coordinates may need an integral?) |

- holonomic constraints, systems
- nonholonomic constraints, systems

8.2 Local, Differential Formulation: D’Alembert’s Principle and Lagrange’s Equations

The physical (and local) concept of virtual-work-less constraint, together with Newton’s second law of motion, generates all of the equations of the Lagrangian and Hamiltonian formulations of classical mechanics. It is required in both the holonomic and nonholonomic equations.

“In practice, the restriction (to virtual-work-less constraints) presents little handicap to the applications, as most problems in which the nonholonomic formalism is used relate to rolling without slipping, where the constraints are obviously workless.”

I give some names to the intermediate equations for organizational and mental-recall purposes, and I put a star (*) by the equations that seem to be most fundamental or important to remember:
<table>
<thead>
<tr>
<th><strong>D’Alembert’s Principle</strong></th>
<th>Virtual-work-less constraining forces $\sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ and $N2 F - \dot{p} = 0$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$&lt;br/&gt; with generalized coordinate and force transformations:&lt;br/&gt; $\Rightarrow \sum_j [Q_j^o - { \frac{d}{dt} (\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} }] \delta q_j = 0$&lt;br/&gt; $\Rightarrow \sum_i (F_i^o - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>“D’Alembert’s Principle with Gnzd. Potential”</strong></td>
<td>D’Alembert’s Principle, with a potential: some applied forces derivable from a generalized potential $U(q_j, \dot{q}_j, t)$&lt;br/&gt; $\Rightarrow \sum_j [Q_j^r - { \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} }] \delta q_j = 0$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>“Monogenic D’Alembert’s Principle”</strong></td>
<td>D’Alembert’s Principle, with monogenic forces: all applied forces derivable from a generalized potential $U(q_j, \dot{q}_j, t)$&lt;br/&gt; $\Rightarrow \sum_j \left[ \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} \right] \delta q_j = 0$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>“Holonomic D’Alembert’s Principle”</strong></td>
<td>D’Alembert’s Principle, with holonomic constraints: (* independent $q_j$’s)&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j^o$&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>“Generalized Lagrange’s Eqn”</strong></td>
<td>D’Alembert’s Principle, with holonomic constraints and a potential: some applied forces derivable from a generalized potential $U(q_j, \dot{q}_j, t)$&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j^r$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>“Monogenic Generalized Lagrange’s Eqn”</strong></td>
<td>D’Alembert’s Principle, with holonomic constraints and monogenic forces: all applied forces derivable from a generalized potential $U(q_j, \dot{q}_j, t)$&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j^r$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>Lagrange’s Eqn</strong></td>
<td>D’Alembert’s Principle, with holonomic constraints and monogenic conservative forces: all applied forces derivable from a conservative potential $U(q_j)$&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>“Dissipative Generalized Lagrange’s Eqn”</strong></td>
<td>Generalized Lagrange’s Equation, with nonconservative forces derivable from a dissipation function $F_i^d = -\frac{\partial \mathcal{F}}{\partial \dot{v}_i}$, so $Q_j^d = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = Q_j^n$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
<tr>
<td><strong>Nonholonomic Lagrange’s Eqn</strong></td>
<td>Monogenic D’Alembert’s Principle, with special nonholonomic eqns of constraint $\sum_j a_{kj} dq_j + a_{kt} dt = 0$&lt;br/&gt; and usage of the method of Lagrange (undetermined) multipliers:&lt;br/&gt; $\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}<em>j} \right) - \frac{\partial L}{\partial q_j} = \sum_k \lambda_k a</em>{kj} (= Q_j^c) \quad j \in \mathbb{N}<em>n$&lt;br/&gt; $\Rightarrow \sum_j a</em>{kj} \dot{q}<em>j + a</em>{kt} = 0 \quad k \in \mathbb{N}_m$&lt;br/&gt; $\Rightarrow \sum_i (F_i^0 - \dot{p}_i) \cdot \delta r = 0$ *&lt;br/&gt;</td>
</tr>
</tbody>
</table>

The “Dissipative Generalized Lagrange’s Equation” is the most generalized, concept-packed *holonomic* equation, so it should definitely be memorized. The Nonholonomic Lagrange’s Equation is used less frequently but should be kept in mind.
\[ F = F^c + F^o \]
\[ F^o = F^p + F^r \]
\[ F^r = F^d + F^n \]

“c” for constraint, “o” for other, “p” for derivable from a generalized potential, “r” for remaining, “d” for derivable from a dissipation function, “n” for not derivable (are there any such things, or might they be derivable from something else?)

\[ Q = Q^c + Q^o = Q^c + Q^p + Q^r = Q^c + Q^p + Q^d + Q^n \]

(The usual holonomic Lagrangian formalism gets rid of the constraining forces.)

- Kinetic energy

\[ T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left( \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right)^2 \]
\[ = M_0 + \sum_j M_j \ddot{q}_j + \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k \]
\[ = T_0 + T_1 + T_2 \]

- Generalized force \( Q_j \) (conjugate? to the generalized coordinate \( q_j \))

\[ \dot{Q}_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \]
\[ Q^o_j = \sum_i F^o_i \cdot \frac{\partial r_i}{\partial q_j} \]
\[ Q^p_j = \frac{d}{dt} \left( \frac{\partial U}{\partial q_j} \right) - \frac{\partial U}{\partial q_j} \]

How does one come up with such a potential?

\[ Q^d_j = \sum_i F^d_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i -\frac{\partial F}{\partial v_i} \cdot \frac{\partial r_i}{\partial q_j} = \sum_i -\frac{\partial F}{\partial v_i} \cdot \frac{\partial v_i}{\partial \dot{q}_j} = -\frac{\partial F}{\partial \dot{q}_j} \]

- Lagrangian \( L \)

\[ L(q(t), \dot{q}(t), t) \equiv T(\dot{q}(t)) - U(q(t), \dot{q}(t), t) \]

The dependence of \( L \) on time may arise if the constraints are time-dependent, if the transformation equations connecting the rectangular and generalized coordinates explicitly contain the time, or if the generalized potential is explicitly time-dependent.

### 8.2.1 Freedom of Lagrangian

(Gauge?) freedom of the Lagrangian:

\[ L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \]

While \( L = T - V \) is always a suitable way to construct a Lagrangian for a conservative system, it DOES NOT provide the only Lagrangian suitable for the given system; you may also use \( L' = T - V + d_t F \).
8.3 Comments and Elaboration

- Virtual displacements and work
  - \( \delta \mathbf{r}_i \) is a virtual, infinitesimal displacement of the coordinate \( \mathbf{r}_i \) (at the instant \( t \))
    * the coordinate could be the position of a particle
    * \( \delta \mathbf{r}_i \) is taken to be consistent with the forces and constraints imposed on the system at the given instant \( t \)
    * “virtual” distinguishes this from actual displacements, which occur over some time interval \( dt \), during which the forces and constraints may be changing
  - \( \mathbf{F}_i = \mathbf{F}_i^c + \mathbf{F}_i^o \): the total force (on a particle) is decomposed into the net constraining force and the net “other forces”
    * We restrict ourselves to systems for which the net virtual work of the forces of constraint is zero so that we may neglect them in our analysis: \( \sum_i \mathbf{F}_i^c \cdot \delta \mathbf{r}_i = 0 \)
    * (This is true for rigid bodies and a large number of other constraints.)
    * (This is true for articles constrained to move on a surface, unless there are sliding friction forces.)
      (What about if those sliding friction forces are considered as applied or external?)
      (If the surface is moving in time, the virtual work is still zero even though the actual work over \( dt \) may not be.)
    * (Rolling friction doesn’t defy this condition.)
  - Equilibrium
    * Overall equilibrium: \( \mathbf{F}_i = 0 \), so \( \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \)
    * Our assumption \( \Rightarrow \sum_i \mathbf{F}_i^o \cdot \delta \mathbf{r}_i = 0 \) (This eqn is called the principle of virtual work.)
      The virtual work of the applied forces vanishes.
      In general \( \mathbf{F}_i^o \neq 0 \) since the \( \dot{\delta} \mathbf{r}_i \) are not completely independent but are connected by the constraints

- Generalized forces
  
- Lagrangian: \( L \)
  - Lagrange density or, again, “Lagrangian”: \( \mathcal{L} \) or \( \mathcal{L} \) (which should I use?)
    The Lagrangian is a function on the tangent bundle. (Wikipedia: Differentiable Manifold)

8.4 Nonconservative Forces, Dissipation Functions, and so forth

- Conservative Forces
- Nonconservative Forces
  - Derivable from a Generalized Potential (velocity dependent or otherwise?)
  - Derivable from a Dissipation Function
    \( F^d_i = -\frac{\partial \mathcal{F}}{\partial q_i} \) or \( \mathbf{F}^d = \nabla_v \mathcal{F} \)
    \( Q^d_j = \sum_i F^d_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \nabla_v \mathcal{F} \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial q_j} \)
  - What’s the rest of ‘em?
8.5 Method of Lagrange Multipliers

- Usable whenever the eqns of constraint are in the form

\[ \sum_j a_{lj} dq_j + a_{lt} dt = 0 \]

and we may take, since virtual displacements \( dq_j = \delta q_j \) imply that \( dt = 0 \),

\[ \sum_k a_{lj} \delta q_j = 0 \quad (\Rightarrow \text{virtual-work-less constraints}) \]

- Includes holonomic constraints. Useful when

(1) it’s convenient to reduce all the \( q \)'s to ind coords
(2) want to solve for the forces of constraint

We get \( n + m \) equations for \( n + m \) unknowns: (the first being Lagrange’s Eqns for nonholonomic systems)

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \sum_k \lambda_k a_{kj} \quad (= Q^c + Q^o ?) \quad j \in \mathbb{N}_n \]

\[ \sum_j a_{kj} \dot{q}_j + a_{kt} = 0 \quad k \in \mathbb{N}_m \]

8.6 Global, Integral Formulation: Hamilton’s Principle and the Euler-Lagrange Equations

8.6.1 Assumptions

- Monogenic systems
- (Virtual-work-less systems?)

<table>
<thead>
<tr>
<th>Hamilton’s Principle</th>
<th>( \delta I \equiv \delta \int_{t_1}^{t_2} dt L = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>The motion of the system from time ( t_1 ) to ( t_2 ) is such that the line integral ( I = \int_{t_1}^{t_2} dt L ), where ( L = T - V ), has a stationary value for the correct path of the motion.</td>
</tr>
<tr>
<td></td>
<td>I.e., the motion is such that the variation of the line integral ( I ) for fixed ( t_1 ) and ( t_2 ) is zero:</td>
</tr>
<tr>
<td></td>
<td>( \delta I = \delta \int_{t_1}^{t_2} dt L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0 )</td>
</tr>
<tr>
<td></td>
<td>(Can extend this to include some nonholonomic systems, but this formulation is most useful for holonomic systems, wherein a Lagrangian of independent coordinates can be set up.)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Euler-Lagrange Eqns</th>
<th>( \frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \dot{y}_i \equiv \frac{dy_i}{dx}; i \in {1, 2, \ldots, n} )</td>
</tr>
<tr>
<td></td>
<td>( x \rightarrow t, y_i \rightarrow q_i, f(y_i, \dot{y}_i, x) \rightarrow L(q_i, \dot{q}_i, t) )</td>
</tr>
<tr>
<td></td>
<td>Extensions: Consider ( f = f(y_i, \dot{y}_i, \ddot{y}_i, \ldots, x) ), or use several parameters ( x_j ), yielding a multiple integral and derivatives of each ( y_i ) with respect to each ( x_j ), and/or consider variations in which the end points are not held fixed.</td>
</tr>
</tbody>
</table>
• (see Goldstein pg 48: “In this dress, Hamilton’s principle says... \( \delta \int_{t_1}^{t_2} dt T = -\int_{t_1}^{t_2} \sum_k Q_k \delta q_k dt \))
• “The variational principle formulation has been justly described as ‘elegant,’ for in the compact Hamilton’s principle is contained all of the mechanics of holonomic systems with forces derivable from potentials. The principle has the further merit that it involves only physical quantities that can be defined without reference to a particular set of generalized coordinates, namely, the kinetic and potential energies.”
• Lagrange’s eqns \(\iff\) Hamilton’s principle
• (Hamilton’s principle \(\implies\) Lagrange’s eqns) is more important than the converse (why? since Hamilton’s principle yields more? Like what?)
9 Hamiltonian Formalism

9.1 Legendre Transformations, Variational Principles, and Hamilton’s Equations

<table>
<thead>
<tr>
<th>Original Function</th>
<th>Legendre Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(x)$</td>
<td>$\psi(p)$</td>
</tr>
<tr>
<td>A collection of points in the $xy$-plane</td>
<td>As a description of the original function:</td>
</tr>
<tr>
<td>As a description of the Legendre transform:</td>
<td>a collection of tangent lines of slope $p$ and</td>
</tr>
<tr>
<td>a collection of tangent lines of slope $-x$ and $\psi$-intercept $\psi$</td>
<td>$y$-intercept $\psi$</td>
</tr>
<tr>
<td></td>
<td>A collection of points in the $p\psi$-plane</td>
</tr>
</tbody>
</table>

- **Original Function**
  - Domain: $\mathbb{R}^n$
  - Codomain: $\mathbb{R}$ (for convenience, imagine each value as a different color)

- **Original Function Level Subspaces** (generalization of level curves)
  - 1D domain: 1-2 points (one color)
  - 2D domain: monochromatic curves
  - 3D domain: monochromatic surfaces

- **Tangent Objects**
  - 1D domain: line (hyperpoint of constant color-velocity)
  - 2D domain: cone/pyramid (hyper-(closed curve) with continuous distribution of color-velocity that is constant in time)
    - possibly also hyper-(curve segment)
  - 3D domain: hyperbubble (hyper-(closed surface))
    - possibly also hyper-(bubble segment)

- **Osculating Subspace for Original Function and Tangent Object**
  - 1D domain: 1 point (one color)
  - 2D domain: polychromatic curves
  - 3D domain: polychromatic surfaces
Energy function $h$

$$h(q, \dot{q}, t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L(q, \dot{q}, t)$$

- The energy function $h$ is identical in value with the Hamiltonian $H$ but is a function of $n + 1$ independent variables (the $q$’s plus $t$) and the $\dot{q}$’s (which depend on the $q$’s) while $H$ is a function of $2n + 1$ independent variables.
- $h$ is not necessarily the mechanical energy, and it is not necessarily conserved.
- Whereas the Lagrangian is uniquely fixed for each system by the prescription $L = T - U$, independent of the choice of generalized coordinates, the energy function $h$ depends in magnitude and functional form on the specific set of generalized coordinates. For one and the same system, various energy functions $h$ of different physical content can be generated depending on how the generalized coordinates are chosen.
- If frictional forces are present and derivable from a dissipation function $F$, then $F$ is related to the decay of $h$.

### Initial Definition of Canonical Momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Also called constitutive relations.

### from Lagrange’s (monogenic, nonrel.) Equation

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

### Initial Definition of the Hamiltonian

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$$

with summation over $i$ and $\dot{q} = \dot{q}(q, p, t)$

### Modified Hamilton’s Principle

$$\delta I = \int_{t_1}^{t_2} \dot{q}_i \dot{p}_i - H(q, p, t) = 0$$

### Hamilton’s (Canonical) Equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad -\dot{p}_i = \frac{\partial H}{\partial q_i}$$

### Hamilton’s Equations in Matrix/Symplectic Notation

$$\dot{\eta} = J \frac{\partial H}{\partial \eta}$$

- $\eta$ is the canonical variable vector:
  
  $$\eta_i = q_i, \ \eta_{n+i} = p_i; \ i \leq n$$

- $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where $0$ and $I$ are $n \times n$
  
  - (Properties of the matrix $J$ here)

- $\left( \frac{\partial H}{\partial \eta} \right)_i = \frac{\partial H}{\partial \eta_i}$

About the symplectic notation: “Considerable ingenuity has been exercised in devising nomenclature schemes that result in entirely symmetric equations, or combine the two sets into one. Most of these schemes have only oddity value, but one (the symplectic notation) has proved to be an elegant and powerful tool for manipulating the canonical equations and allied expressions.”

### 9.2 Constructing the Hamiltonian via the Lagrangian Formulation

1. With a chosen set of generalized coordinates, $q_i$, the Lagrangian $L(q_i, \dot{q}_i, t)$ is constructed.

2. The conjugate momenta are defined as functions of $q_i$, $\dot{q}_i$, and $t$ by the (initial) definition of $p_i = \frac{\partial L}{\partial \dot{q}_i}$. 

17
3. The Hamiltonian is formed using $H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$. At this stage one has $h$ instead of $H$, or rather some mixed function of $q_i$, $\dot{q}_i$, $p_i$, and $t$.

4. The equations $p_i = \partial L / \partial \dot{q}_i$ are then inverted to obtain $\dot{q}_i$ as functions of $(q, p, t)$. Possible difficulties in the inversion SHOULD BE ADDRESSED SOMEWHERE.

5. The results of the previous step are then applied to eliminate $\dot{q}$ from $H$ so as to express it solely as a function of $(q, p, t)$

9.3 Variational Principles

- action, abbreviated action, (Maupertuis action)
- Variational Principles of Mechanics

“A host of similar variational principles for classical mechanics can be derived in bewildering variety. The variational principles in themselves contain no new physical content (with respect to Newton’s Laws or Lagrange’s equations), and they rarely simplify the practical solution of a given mechanical problem. Their value lies chiefly as starting points for new formulations of the theoretical structure of classical mechanics. For this purpose Hamilton’s principle is especially fruitful, and to a lesser extent, so also is the principle of least action. The others have proved to be of little use, except as they have led to fruitless teleological speculations (on final cause or purpose and design).”

- Principle of Stationary Action (“Least Action”)
- Jacobi’s for of the least action principle
- Hertz’s principle of least curvature
- Fermat’s principle in geometrical optics (of least time for light)
- (Principle of maximal proper time in general relativity)

9.4 Canonical Transformations

- Passive
- Active

10 Alternative Formulations

- Routhian
  ~ “… Routh’s is most useful in engineering applications. But the Routhian is a sterile hybrid of the Lagrangian and Hamiltonian pictures; for developing various formalisms of classical mechanics, the complete Hamiltonian formulation is more fruitful.”
- Various others
11 Example Lagrangians, Hamiltonians, and Equations of Motion

<table>
<thead>
<tr>
<th>Nonrelativistic charged particle in electromagnetic fields (mass (m), charge (q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ U = q(\Phi - \mathbf{A} \cdot \mathbf{v}) ]</td>
</tr>
<tr>
<td>[ L = \frac{1}{2} m \dot{\mathbf{r}}^2 - q\Phi + q\mathbf{A} \cdot \mathbf{v} ]</td>
</tr>
<tr>
<td>[ H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\Phi ]</td>
</tr>
</tbody>
</table>

- The Hamiltonian is **not** \(T + U\), but it is the total energy (WHY?)

**Some other situation**

**When a Hamiltonian Equals Total Energy**

- The Lagrangian is the sum of functions each homogeneous in the generalized velocities of degree 0, 1, and 2, respectively. So \(H = L_2 - L_0\) (cf. Eq. 2-57 in Goldstein).
- The equations defining the generalized coordinates don’t depend on time explicitly, so \(L_2 = T\).
- The forces are derivable from a conservative potential \(V\), so \(L_0 = -V\) and \(T + V = E\).

\[ H = T + V = E \]

**Hamiltonian when Lagrangian has Simple Form**

- \(L = L_0 + L_1 + L_2\)
  \[ L(q, \dot{q}, t) = L_0(q, t) + k^T \dot{q} + \frac{1}{2} \dot{q}^T T \dot{q} \]
  - \(T\) is a symmetric matrix (and what about the positive definite property of the kinetic energy? see below)
- \(\mathbf{p} = T \dot{q} + k\)
  - \(\dot{q} = T^{-1}(\mathbf{p} - k)\)
  - \(T^{-1}\) normally exists by virtue of the positive definite property of the kinetic energy
- \(h = \frac{1}{2} \dot{q}^T T \dot{q} - L_0\)
  \[ H(q, p, t) = \frac{1}{2}(p - k)^T T^{-1}(p - k) - L_0(q, t) \]

**Something else**

12 Ch 8

- Legendre Transformations and the Hamilton Equations of Motion
  When coming up with the Hamiltonian formulation, we say
  \[ p_j = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_j} \]
but when using the Hamiltonian formulation we assume $p_j$ is independent of $q_j$. Can we recover this equation? If so, is it different in this situation because it’s an equation of motion rather than a definition?

- Ignorable Coordinates and Conservation Theorems
- Routh’s Procedure and Oscillations about Steady Motion
- The Hamiltonian Formulation of Relativistic Mechanics
- Derivation of Hamilton’s Equations from a Variational Principle
- The Principle of Least Action

13 Conservation Theorems and Symmetry Properties

13.1 Solving the Equations of Motion

- A system of $n$ degrees of freedom will have $n$ differential eqns that are second order in time.
- The solution of each eqn will require two integrations resulting in $2n$ constants of integration.
  - Could be the initial conditions (the $n$ $q_j$’s and the $n$ $\dot{q}_j$’s at the initial time $t_i$.)
- Canonical or Conjugate Momentum (generalized momentum conjugate to the generalized coordinate $q_j$)

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

If $q_j$ is not Cartesian, then $p_j$ will not necessarily have the dimensions of a linear momentum. When $q_j$ is Cartesian, if the potential is velocity-dependent, then the generalized momentum will not be identical with the usual mechanical momentum.

- If a coordinate is ignorable (the Lagrangian is independent of that coordinate), then its conjugate momentum is conserved, or constant. This is more general than the Newtonian momentum conservation laws (which require WN3 or SN3). E.g., for a charged particle in an electromagnetic field,

$$p_x = m\dot{x} + qAx$$

is conserved if $\Phi$ and $A$ are independent of $x$. ($qAx$ is the electromagnetic linear momentum associated with the field of the particle)

- energy function

$$h(q, \dot{q}, t) = \sum_j q_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

- Euler’s theorem, $\ldots$, $h = T + V$ in some cases but not others, conserved in some cases but not others, $\ldots$, dissipation function

$$\frac{dh}{dt} = -2\mathcal{F} - \frac{\partial L}{\partial t}$$

sometimes:

$$\frac{dE}{dt} = -2\mathcal{F}$$
14 Theorems

- **Liouville’s Theorem** and Equation (of classical statistical and Hamiltonian mechanics)

  The Liouville equation describes the time evolution of phase space measure (a.k.a. distribution function in physics).

  \[
  \frac{d}{dt} \rho = \nabla \cdot (\dot{q} \rho + \rho \dot{p}) = 0
  \]

  \[
  \frac{\partial\rho}{\partial t} = -\{\rho, H\}
  \]

  \[
  (\partial_t + \hat{L})\rho = 0
  \]

  where

  \[
  \hat{L} \equiv \sum_{i=1}^{d} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)
  \]

  is the Liouvillian or Liouville operator.

  - Related to symplectic topology, ergodic theory, Hamiltonian flow, conservation (Noether’s Thm) etc.
  - The equation is valid for both equilibrium and nonequilibrium systems. It is a fundamental equation of nonequilibrium statistical mechanics. Generalized to collisional systems it is called the Boltzmann equation.
  - The equation is integral to the proof of the fluctuation theorem from which the second law of thermodynamics can be derived. It is also the key component of the derivation of Green-Kubo relations for linear transport coefficients such as shear viscosity, thermal conductivity or electrical conductivity.
  - Virtually any textbook on Hamiltonian mechanics, advanced statistical mechanics, or symplectic geometry will derive the Liouville theorem.
  - Quantum analogues: Lindblad equation for the density matrix \( \rho \), Ehrenfest theorem (each missing the last piece?)

  \[
  \frac{\partial_t \rho}{\partial t} = -\frac{1}{i \hbar} \{\rho, H\}
  \]

  \[
  \left( \frac{d_t \rho}{\partial t} = -\frac{1}{i \hbar} [\rho, H] - \frac{1}{\hbar} \sum_{n,m} h_{n,m} (\rho L_m L_n + L_m L_n \rho - 2 L_m \rho L_n) + \text{h.c.} \right)
  \]

  \[
  \frac{d_t}{\partial t} \langle A \rangle = \frac{1}{i \hbar} \langle [A, H] \rangle (\langle \rho A \rangle)
  \]

  from [http://en.wikipedia.org/wiki/Liouville’s_theorem_(Hamiltonian)]... why are the last pieces not included?

  The sign difference follows from the assumption that the operator is stationary and the state is time-dependent.

- **Helmholtz Theorem** (of classical mechanics)

  This theorem should be distinguished from Helmholtz’s theorem in vector calculus (the fundamental theorem of vector calculus, relating to Helmholtz decomposition) and Helmholtz’s theorems in fluid mechanics, all named after Hermann von Helmholtz.

- **Generalized Helmholtz Theorem**
• (1873) **Bertrand’s Theorem** of central force, two-body motion
The only central forces that result in closed orbits for all bound particles are the inverse square law and Hooke’s law.

• General characteristics ⇒ quantitative information
  - The degenerate character of orbits in a gravitational field fixes the form of the force law.
  - Degenerate means that the periods of oscillation of the body in two variables \((r, \theta)\) are commensurate, so that the orbit is closed.
  - (It is not necessary to use the elliptic character of the orbits to arrive at the gravitational force law.)
  - (“Later on we shall encounter other formulations of the relation between degeneracy and the nature of the potential.”)

• **Larmor’s Theorem**
To first order in \(B\), the effect of a constant magnetic field on a classical system is to superimpose on its normal motion a uniform precession with angular frequency \(\omega_l = -qB/2m\) (the Larmor frequency).

• **Noether’s Theorem**
Has a Lagrangian formulation and a Hamiltonian formulation.
Relates symmetries and conservation laws.
(Is this what I start with? [See Peskin & Schroeder pg 17] Given that the action is invariant under a transformation \(\phi_1 = e^{ia_\beta j_\beta} \phi\), or, equivalently, given that the Lagrangian density is invariant up to a four-divergence under the transformation, we have)

\[
(\mathcal{L}(x) = \mathcal{L}(x) + \delta \mathcal{L}(x))
\]

(where \(\delta \mathcal{L}(x) = \delta a_j \partial_\mu J^\mu_j(x)\))

\[
S = \int d^4x \mathcal{L}(\phi_\alpha, \partial_\mu \phi_\alpha)
\]

\[
0 = \delta S
\]

\[
= \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi_\alpha} \delta \phi_\alpha + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \delta (\partial_\mu \phi_\alpha) \right]
\]

\[
= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\alpha} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \right) \delta \phi_\alpha + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \delta \phi_\alpha \right) \right\}
\]

\[
\Rightarrow \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \delta \phi_\alpha \right) \equiv \partial_\mu j^\mu = 0
\]

\[
\delta \phi_\alpha = i \delta a_j (g_j)_{\alpha \beta} \phi_\beta
\]

\[
\phi_1 = e^{ia_\beta j_\beta} \phi
\]

\[
\mathcal{J}^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \delta \phi_\alpha
\]

\[
= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} (i \delta a_j (g_j)_{\alpha \beta} \phi_\beta)
\]

\[
= i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\alpha)} \delta a_j (g_j)_{\alpha \beta} \phi_\beta
\]

\[
\partial_\mu \mathcal{J}^\mu = 0
\]
but since $\delta a_j$ is arbitrary, we have

\[ j^\mu_k = \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (\delta \phi_\alpha)_k \]
\[ = \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (i (g_k)_{\alpha \beta} \phi_\beta) \]
\[ = i \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (g_k)_{\alpha \beta} \phi_\beta \]

\[ \partial_\mu j^\mu_k = 0 \]

Now we define the physical conserved current and conserved charge:

\[ J^\mu_k \equiv -j^\mu_k \]
\[ = -\frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (\delta \phi_\alpha)_k \]
\[ = -i \frac{\partial L}{\partial (\partial_\mu \phi_\alpha)} (g_k)_{\alpha \beta} \phi_\beta \]

\[ Q_k \equiv \int d^3x \ J^0_k(x) \]
\[ = -i \int d^3x \ \frac{\partial L}{\partial (\partial_0 \phi_\alpha)} (g_k)_{\alpha \beta} \phi_\beta \]

15 Central Forces: Trajectories, Orbits, and Scattering

16 Open Questions and Mysteries
17  Class on Mechanics and Field Theory

17.1 Books to Use

• Goldstein: Classical Mechanics
• Landau and Lifshitz: Mechanics
• Davison Soper: Classical Field Theory
• V.I. Arnold (A. Weinstein, and K. Vogtmann): Mathematical Methods of Classical Mechanics
• Flanders: Differential Forms with Applications to the Physical Sciences
• MTW: Gravitation (has interesting things to say about E&M and perhaps Mechanics)
• Interesting references
  – From Goldstein:
    * F. Klein and A. Sommerfeld, *Theorie des Kreisels*, a monumental work on the theory of the
      top. In Volume II, many pages are given to a thorough demolishing of the popular or elementary
      “derivations” of gyroscopic precession. The authors remark that it was the unsatisfactory nature
      of these derivations that led them to write the treatise!

17.2 Class Topics

Advanced Classical Mechanics and Classical Field Theory (and their relation to all subtopics of physics)
from physical and mathematical perspectives

• Lagrangian and Hamiltonian formalism
  – Lagrangian: generalized coords (and momenta)
  – Hamiltonian: canonical coords (and momenta)
  – Legendre transformations, generating functions (physics versus math generating functions)
  – Canonical transformations (→ contact transformations, symplectomorphisms; Hamilton-Jacobi eqns,
    conserved quantities)
  – Poisson Brackets, Symplectic Manifolds, Hamiltonian flow
  – and the relation to the many quantization schemes

• Theorems in classical mechanics
  – Noether’s Thm
  – Liouville’s Thm, the fluctuation thm and proof for the second law of thermodynamics, Ehrenfest
    thm, Lindblad eqn
  – Helmholtz Thm
  – (Stone-von Neumann thm)?

• Specific Problems:
  – the Runge-Lenz vector and its quantum counterpart
  – Small-distance cuts-off in classical theories (e.g. viscosity, speed of sound, magnetic susceptibility)
    and their relationship to cuts-off in QFT
  – (at atomic scales, where continuum description no longer applies)
    – (see Peskin and Schroeder pg 266)

• Continuum and fluid mechanics
• Dynamical systems: differential equations, bifurcation theory, and Hamiltonian systems; differential
dynamics, including hyperbolic theory and quasiperiodic dynamics; ergodic theory; low-dimensional
dynamics.
  – Chaos and Ergodicity
• Asymptotic methods: Asymptotic Fundamental mathematics of asymptotic analysis, asymptotic ex-
pansions of Fourier integrals, method of stationary phase. Watson lemma, method of steepest descent,
uniform asymptotic expansions, elementary perturbation problems

17.3 Core Topics for Class, from Goldstein
Ch 1 Elementary Principles
Ch 2 Variational Principles and Lagrange’s Eqns
Ch 6 Small Oscillations
Ch 8 Hamilton Eqns of Motion
Ch 9 Canonical Transformations
Ch 12 Lagrangians / Hamiltonians for Fields / Continuous Systems

17.4 Investigate
Ch 10 . . .
Ch 11 . . .
• Landau and Lifshitz Mechanics

17.5 Take note of . . .
• pg ix
  – Manipulation of tensors in non-Euclidean spaces (in Chs. 6 and 7)
  – Complex Minkowski space
• pg xi-xii
  – the technique of action-angle variables is needed for the older quantum mechanics
  – the Hamilton-Jacobi eqn and the principle of least action provide the transition to wave mechanics
  – Poisson brackets and canonical transformations are invaluable in formulating the newer quantum
  mechanics
  – classical mechanics affords the student an opportunity to master many of the mathematical tech-
niques necessary for quantum mechanics while still working in terms of the familiar concepts of
classical physics
  * the discussion of central force motion includes the kinematics of scattering and the classical
  solution of scattering problems
  * . . . by the technique [of a unified mathematical treatment of rotations in terms of the eigen-
value problem for an orthogonal matrix] it becomes possible to include at an early stage the
difficult concepts of reflection operations and pseudotensor quantities . . . [and] “spinors” can be
introduced in connection with the properties of Cayley-Klein parameters.
References

