Notes on Classical Electrodynamics

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Contents

1 Questions and Ideas

Jackson
pg 196 2nd paragraph: Mulitple-valued magnetic scalar potential?!
pg 197

- Should I write d^2x or $\mathrm{d}a$? d^3x or $\mathrm{d}v$? (Or both, depending on aesthetics and circumstance?) Is d^2x more instructive for problem solving than $\mathrm{d}a$, or can it give the wrong idea? (When taking a line integral along an arbitrary curve, $\mathrm{d}x$ gives the wrong idea while $\mathrm{d}s$ is not misleading.)
- Must we start with retarded potentials rather than the retarded fields? (It appears that we could start either way.)
- Why does it matter how much a test charge "disturbs" the electromagnetic field? Shouldn't it still behave the same (so long as it doesn't cause the sources to move)? Or is there some self-interaction factor that comes into play?
- Why does the magnetic dipole moment density definition have $\frac{1}{2}$ in it? (Jackson pf 186)
- Jackson writes $\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega}$ on page 676, but isn't this a contradiction in notation? (Maybe it's alright in non-standard analysis, but usually I would expect that if a function depends on two independent variables, one would have to write partial derivatives by convention:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}\omega\,\mathrm{d}\Omega} \neq \frac{\partial^2 I}{\partial\omega\,\partial\Omega}$$

Ideas

- Among the infinite family of exact superluminal solutions of the Maxwell equations are waves known as **X-waves**.
- Include *p*-form electrodynamics.
- Explain how a battery works.
- Explain triboelectric effects.

Books

What are the best books? For what?

- Griffiths
- Jackson
- Pnofsky and Phillips
- Wangsness
- Franklin (Conceptual, more basic, how to do problems)

2 The Big Picture

Context of Physics

(from static cling and lodestones to electroweak unification), unified electricity and magnetism, QFT

Is this right?:

- A rather sophisticated view: Picture one lone charged particle (and its own field) sitting in an inertial frame in an otherwise empty universe. (Perhaps the concept of an inertial frame in this stark universe is a bit difficult.) This non-accelerating charged particle and it's electromagnetic field can be viewed as a system in dynamical equilibrium, where the particle (or "quantum") is radiating photons, energy, and momentum, while at the same time absorbing the same amount of photons, energy, and momentum. (The photons must be *coming back* to the particle, mustn't they, since there is no source of radiation at infinity radiating photons back to the particle, spherically symmetrically from the view of the particle.)

  - So any charged particle always comes with a busy swarm of photons. (What frequencies do these photons have? What's the distribution?)
  - Might there have been a time when a charged particle did not necessarily come with such a swarm of photons, say near the big bang? Is it possible that they be separated, or are they really just one and the same process-object?

- **Do charged particles in uniform acceleration radiate?**
  If so, then when the charged particle is accelerated, non-equilibrium states are reached where there is energy transfer from the particle to the fields. The particle emits more photons than it absorbs, adding energy and momentum to the fields. The energy in the fields radiates away in electromagnetic waves, which take momentum with them, and the fields exert a recoil force on the particle that causes resistance to the acceleration – this is called radiation reaction.
- Considering the particle and its own fields as a system, this is a "self-force". The particle somehow knows it's accelerating and radiates more photons which act to reduce its acceleration. (A kind of Lenz's law.)
  - How does (or doesn't) the radiation pattern relate to length contraction? How does the pattern of (virtual?) photon radiation/absorption change with acceleration (or jerk)?
  - What about runaway solutions and acausal acceleration?
- How can spin be explained in this picture? What are the photons doing?

Domains

If one were exacting, one might divide the topic into the following categories:

Electromagnetism or **Electromagnetics**	
Electrokinematics	Magnetokinematics
Electromagnetokinematics	
Electrostatics	Magnetostatics
Electromagnetostatics	
Electrodynamics	Magnetodynamics
Electromagnetodynamics	

This might be a natural division if there was such a thing as magnetic charge. Instead, the topic is divided into the following categories:

Electrodynamics	
Electrostatics	Magnetostatics
Electrodynamics	

In electrostatics, charges are stationary and fields constant in time, in magnetostatics, fields are constant in time but charges can move, and in electrodynamics, charges can move and fields can change in time. We could also call Magnetostatics "Steady Currents" or "Steady-State Electrodynamics". We might also include the categories of quasi-statics (quasi-electrostatics) and quasi-static fields (quasi-magnetostatics).

- Classical versus Quantum
  - In what ways are the classical results correct, and where/how are they wrong?

- Non-relativistic versus Relativistic (for matter that interacts with the fields)

Historical Developments

- 1831 - Macedonio Melloni demonstrates that black body radiation can be reflected, refracted, and polarised in the same way as light

3 Notation and Convention

Notation in These Notes

- $\mathbf{x} \equiv \mathbf{r}$ is the observation point or "test" point ($\mathbf{x} = \mathbf{r} = r\hat{\mathbf{r}}$; the circumflex denotes a unit vector)
  - Usually, for any vector $\mathbf{v}$, $\mathbf{v} = v\hat{\mathbf{v}}$, but note that $\mathbf{x} \neq x\hat{\mathbf{x}}$. (One could adopt $\mathbf{x} \equiv x\hat{\mathbf{x}}$, so $\mathbf{x} \neq \mathbf{r}$)
  - (Maybe change notes to adopt $\mathbf{x} = x\hat{\mathbf{x}}$, and note two uses of s,
    cylindrical radius s ($\mathbf{s} = s\hat{\mathbf{s}}$ and $\mathrm{d}\mathbf{s} = \mathrm{d}s\hat{\mathbf{s}}$ where $\hat{\mathbf{s}}$ is location-dependent, $\mathrm{d}\mathbf{s} \neq \mathrm{d}\mathbf{r}$) and
    path-length coordinate s ($\mathbf{s}$ not defined[1] and $\mathrm{d}\mathbf{s} = \mathrm{d}s\hat{\mathbf{s}}$ where $\hat{\mathbf{s}}$ is location-dependent, $\mathrm{d}\mathbf{s} = \mathrm{d}\mathbf{r}$),
    $\mathrm{d}\mathbf{r} \neq \mathrm{d}r\hat{\mathbf{r}}$)

    It may be best to keep $\mathbf{x} \neq x\hat{\mathbf{x}}$ to generalize to the nD case $\mathbf{x} = x^i\hat{\mathbf{x}}_i$, which is convenient notation
- $\mathbf{x}' \equiv \mathbf{r}'$ is a source point, usually a variable of integration over some region ($\mathbf{x}' = \mathbf{r}' = r'\hat{\mathbf{r}}'$)
  - in these notes, using $\mathbf{x}$ ($\mathbf{x}'$) is usually meant to emphasize rectilinear geometry (integration), while $\mathbf{r}$ ($\mathbf{r}'$) is usually meant to emphasize spherical or arbitrary geometry (integration)
- $\mathbf{R} \equiv \mathbf{x} - \mathbf{x}'$ is a relative vector ($\mathbf{R} = R\hat{\mathbf{R}} \equiv R\hat{\mathbf{n}}$)
- $\hat{\mathbf{n}}$ will represent a unit vector normal to a surface in addition to the direction of the relative vector $\mathbf{R}$. The meaning should be clear by the context, although I'll try to explicitly confront any ambiguities.
- n is an index of refraction.
- $\underset{\sim}{n}$ is a complex index of refraction. The under-tilde indicates complex-ness, as in $\underset{\sim}{\varepsilon}_{\mathrm{r}}$, a complex relative permittivity, or $\underset{\sim}{\mathbf{k}}$, a complex wave vector.
- t' is a retarded time (a time in the past, perhaps when a source was located at some position $\mathbf{x}'$)
- $\tau \equiv t - t'$ is a relative duration
- In electrodynamics, the letters E, L, V, and U usually denote (and denote in these notes) an electric field magnitude, an inductance, an electrostatic (conservative) electric potential, and an energy. In classical mechanics, these same letters usually denote other quantities, so we'll use calligraphic versions of the last three letters for these other meanings: $\mathcal{L}$, $\mathcal{V}$, and $\mathcal{U}$ denote a Lagrangian, a conservative potential, and a generalized potential. ($\mathcal{E}$ will denote emf.)

Conventional But Defied Notation and Nomenclature

- In relativity and particle kinematics it is usual that the symbols $\{p, E, m, v\}$ stand for $\{pc, E, mc^2, v/c\}$, as when using natural units (or the natural unit procedure), so the equations $E^2 = \mathbf{p}^2 + m^2$ and $\mathbf{v} = \mathbf{p}/E$ are common. I, however, won't use this notation because I prefer dimensionally correct statements and equations that carry as much information as possible. I will also not follow Jackson's convention of using Gaussian units for relativistic expressions. I will always try to use SI units, and will eventually convert all expressions in these notes into SI.

[1] One might define $\mathbf{s}$ to be the net path-displacement $\int \mathrm{d}\mathbf{s}$, but then one might become confused by thinking s is the net distance $|\mathbf{s}|$ instead of the path-length coordinate that yields path-length $|\Delta s| = \left|\int \mathrm{d}s\right|$ or path-length displacement $\Delta s = \int \mathrm{d}s$. Instead, I'll use $\mathbf{d}$ or $\Delta\mathbf{r}$ or something else to designate the net path-displacement.

- I do not use the possibly misleading dichotomic nomenclature "free" and "bound" with regard to charge, and the division of current densities into "free", "bound", and "polarization" current densities. See the entries for Charge and (Effective) Current in the Terms and Quantities section for more information.

Conventional Terms and Phrases

- "Point Charge" - This is really short for the more accurate phrase "point particle with charge", since charge is a property that a particle has. (Similarly, when people speak of manipulating or observing "a mass", they mean "a massive object" or "an object with mass".)
- ...
- See §5 for ambiguous terms and the decided use of them.

Abbreviations in These Notes

- EM = electromagnetic or electromagnetism
- BC = boundary condition
- Thm = theorem

4 Mathematics

- Complex vector notation, equations, and conventions...

  $\underset{\sim}{\mathbf{v}} = \mathbf{v}_\mathrm{r} + i\mathbf{v}_\mathrm{i}$

  $\mathbf{v} \equiv \mathrm{Re}\,\underset{\sim}{\mathbf{v}} = v\,\hat{\mathbf{v}}$

  $\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}}^* = (\mathbf{v}_\mathrm{r} + i\mathbf{v}_\mathrm{i}) \cdot (\mathbf{v}_\mathrm{r} - i\mathbf{v}_\mathrm{i}) = {\mathbf{v}_\mathrm{r}}^2 + {\mathbf{v}_\mathrm{i}}^2$

  – $\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}}^* = 0 \quad \Rightarrow \quad \underset{\sim}{\mathbf{v}} = \mathbf{0}$

  $\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}}^* = 1 \quad \Rightarrow \quad v_\mathrm{r} = \cos\theta$ and $v_\mathrm{i} = \sin\theta$ so $\underset{\sim}{\mathbf{v}} = \cos\theta\,\hat{\mathbf{v}}_\mathrm{r} + i\sin\theta\,\hat{\mathbf{v}}_\mathrm{i}$

  $\underset{\sim}{\mathbf{v}}^2 = \underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}} = (\mathbf{v}_\mathrm{r} + i\mathbf{v}_\mathrm{i}) \cdot (\mathbf{v}_\mathrm{r} + i\mathbf{v}_\mathrm{i}) = \big({\mathbf{v}_\mathrm{r}}^2 - {\mathbf{v}_\mathrm{i}}^2\big) + i\big(2\mathbf{v}_\mathrm{r} \cdot \mathbf{v}_\mathrm{i}\big)$

  – $\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}} = 0 \quad \Rightarrow \quad v_\mathrm{r} = v_\mathrm{i}$ and $\mathbf{v}_\mathrm{r} \perp \mathbf{v}_\mathrm{i}$

  $\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}} = 1 \quad \Rightarrow \quad v_\mathrm{r} = \cosh\theta$, $v_\mathrm{i} = \sinh\theta$, and $\mathbf{v}_\mathrm{r} \perp \mathbf{v}_\mathrm{i}$ so $\underset{\sim}{\mathbf{v}} = \cosh\theta\,\hat{\mathbf{v}}_\mathrm{r} + i\sinh\theta\,\hat{\mathbf{v}}_\mathrm{i}$

Conventions for Complex Vector Magnitudes and Complex Unit Vectors			
Convention 1	**Convention 2**	**Convention 3**	**Convention 4**
$\underset{\sim}{\mathbf{v}} = \lVert\underset{\sim}{\mathbf{v}}\rVert_1\,\hat{\underset{\sim}{\mathbf{v}}}_1$ where $\lVert\underset{\sim}{\mathbf{v}}\rVert_1 \equiv \sqrt{\underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}}^*}$ $= \sqrt{{\mathbf{v}_\mathrm{r}}^2 + {\mathbf{v}_\mathrm{i}}^2}$ so $\hat{\underset{\sim}{\mathbf{v}}}_1 \cdot \hat{\underset{\sim}{\mathbf{v}}}_1^* = 1$	$\underset{\sim}{\mathbf{v}} = \lVert\underset{\sim}{\mathbf{v}}\rVert_2\,\hat{\underset{\sim}{\mathbf{v}}}_2$ where $\lVert\underset{\sim}{\mathbf{v}}\rVert_2 \equiv \sqrt{\lvert\underset{\sim}{\mathbf{v}}^2\rvert}$ $= \sqrt{\big({\mathbf{v}_\mathrm{r}}^2 - {\mathbf{v}_\mathrm{i}}^2\big)^2 + \big(2\mathbf{v}_\mathrm{r} \cdot \mathbf{v}_\mathrm{i}\big)^2}$ so $\lvert{\hat{\underset{\sim}{\mathbf{v}}}_2}^2\rvert = 1$	$\underset{\sim}{\mathbf{v}} = \underset{\sim}{v}\,\hat{\underset{\sim}{\mathbf{v}}}_3$ where $\underset{\sim}{v} = \sqrt{\underset{\sim}{\mathbf{v}}^2}$ (with the smaller of the two possible phases) so ${\hat{\underset{\sim}{\mathbf{v}}}_3}^2 = 1$	$\underset{\sim}{\mathbf{v}} = \underset{\sim}{v}\,\hat{\underset{\sim}{\mathbf{v}}}_4$ where $\underset{\sim}{v} = \sqrt{\underset{\sim}{\mathbf{v}}^2}$ (with the larger of the two possible phases) so ${\hat{\underset{\sim}{\mathbf{v}}}_4}^2 = 1$

$|\underset{\sim}{\mathbf{v}}|$ seems not well-definable: the naturally expected definition is dependent upon the coordinate system

$$\begin{aligned}\left\langle |\underset{\sim}{\mathbf{v}}_x|\,,\left|\underset{\sim}{\mathbf{v}}_y\right|,|\underset{\sim}{\mathbf{v}}_z|\right\rangle_{\mathrm{d}} &\neq \left\langle |\underset{\sim}{\mathbf{v}}_r|\,,|\underset{\sim}{\mathbf{v}}_\theta|\,,|\underset{\sim}{\mathbf{v}}_\phi|\right\rangle_{\mathrm{s}} \\ \left\langle |\mathbf{v}_{\mathrm{r}x}+i\mathbf{v}_{\mathrm{i}x}|\,,\left|\mathbf{v}_{\mathrm{r}y}+i\mathbf{v}_{\mathrm{i}y}\right|,|\mathbf{v}_{\mathrm{r}z}+i\mathbf{v}_{\mathrm{i}z}|\right\rangle_{\mathrm{d}} &\neq \left\langle |\mathbf{v}_{\mathrm{r}r}+i\mathbf{v}_{\mathrm{i}r}|\,,|\mathbf{v}_{\mathrm{r}\theta}+i\mathbf{v}_{\mathrm{i}\theta}|\,,\left|\mathbf{v}_{\mathrm{r}\phi}+i\mathbf{v}_{\mathrm{i}\phi}\right|\right\rangle_{\mathrm{s}}\end{aligned}$$

$\hat{\underset{\sim}{\mathbf{v}}} \equiv \underset{\sim}{\mathbf{u}} = \mathbf{u}_{\mathrm{r}} + i\mathbf{u}_{\mathrm{i}}$

$\|\underset{\sim}{\mathbf{v}}\|_1 = 1$
$\Rightarrow \underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}}^* = 1$

$\|\underset{\sim}{\mathbf{v}}\|_2 = 1$
$\Rightarrow \left(\mathbf{v}_{\mathrm{r}}{}^2 - \mathbf{v}_{\mathrm{i}}{}^2\right)^2 + \left(2\mathbf{v}_{\mathrm{r}} \cdot \mathbf{v}_{\mathrm{i}}\right)^2 = 1$
$\Rightarrow \left(v_{\mathrm{r}}{}^2 - v_{\mathrm{i}}{}^2\right)^2 + \left(2v_{\mathrm{r}}v_{\mathrm{i}}\cos\theta_{\mathrm{ri}}\right)^2 = 1$
$\Rightarrow v_{\mathrm{r}}{}^4 - 2v_{\mathrm{r}}{}^2v_{\mathrm{i}}{}^2 + v_{\mathrm{i}}{}^4 + 4v_{\mathrm{r}}{}^2v_{\mathrm{i}}{}^2\cos^2\theta_{\mathrm{ri}} = 1$
$\Rightarrow v_{\mathrm{r}}{}^4 - 2v_{\mathrm{r}}{}^2v_{\mathrm{i}}{}^2(1 - 2\cos^2\theta_{\mathrm{ri}}) + v_{\mathrm{i}}{}^4 = 1$
$\Rightarrow$ who knows?
$\not\Rightarrow \underset{\sim}{\mathbf{v}} \cdot \underset{\sim}{\mathbf{v}} = 1$

- Vector product identities (3)
- Gradient, Divergence, Curl
- Derivative Product Rules: scalar derivative, gradient (explicit and ang mom forms), div (2), curl (2)
- Derivative Chain Rules: (?) $\boldsymbol{\nabla}' \cdot \mathbf{F}(\mathbf{R}) = \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{r})$ multiplied by something (see Dirac deltas below, where $\mathbf{F}(\mathbf{r}) = \hat{\mathbf{n}}/R^2$)
- Second Derivatives:
  gradients have no curl
  curls have no divergence
  scalar and vector Laplacian, (Green's Identities)
- Generalized Stokes Thm (Fundamental Thm of Manifold-Form Calc) $\Rightarrow$
  Helmholtz Thm (Fundamental Thm of Vector Calc)
  Kelvin-Stokes Curl Thm (gradient, "curl" corollaries), Green Thm
  Ostrogradsky-Gauss Div Thm (gradient, curl corollaries), Green Identities
  (See Jackson inside cover and Griffiths pg 56)
- Linear Second-Order PDEs and BCs (explicated after Ponderable Media):
  - Laplace Eqn (elliptic PDE)
    (w/o sources $\Rightarrow$ scalar potential $\nabla^2\Phi = 0$)
    Given BCs, solns are series of harmonic functions (twice continuously differentiable)
    Cartesian coords: simple harmonic functions ($\sin, \cos, e^{ix}$), hyperbolic sin cos
    Cylindrical coords: for s, (modified) Bessel functions; for θ, z, same as Cartesian
    (Neumann, Hankel funcs; Fourier-Bessel, Neumann, Kapteyn, Schlömilch series)
    Spherical coords: for r, associated Legendre functions; for θ, ϕ, spherical harmonics
    (or Legendre function of the first kind of order ν, a hypergeometric func;
    or Laguerre, Hermite polynomials)
  - Poisson Eqn (elliptic PDE)
    (w/ sources $\Rightarrow$ electric scalar potential $\nabla^2\Phi = -\rho/\varepsilon_0$)

Given Dirichlet or Neumann BCs, Green functions will help solve eqn
(Expansions of Green fuctions in cylindrical and spherical coords)
General soln = complementary (homogeneous) foln (of Laplace eqn) + particular soln

- Helmholtz Eqn (elliptic PDE)
$(\nabla^2 + k^2)u(\mathbf{x}, t) = 0$
2D Cartesian coords: sin, cos
2D polar coords: for s, (modified) Bessel functions; for ϕ, sin, cos
(hyperbolic sin cos) (Hankel functions) (use of Green functions [See Eqn World])
(spherical Bessel functions and spherical harmonics [See Wikipedia])
- Wave Eqn (hyperbolic PDE)
$\Box^2 u = (\nabla^2 - \frac{1}{c}\partial_t{}^2)u(\mathbf{x}, t) = 0$, $u \to \Phi, \mathbf{A}$
Separation of space/time variables $\Rightarrow$ Helmholtz Eqn
Planewave solns
Guage transformations (Lorentz, Coulomb, etc.)
Green functions for wave eqn
Retarded solns for potential
Jefimenko's generalizations of Coulomb/Biot-Savart Laws/Heaviside-Feynman Expressions for pt charge fields

- More Eqns?
- Transformation properties
  - Invariance (Covariance)
  under rotations, Lorentz transf, Poincarè transf, general coord transf, (etc?)
  Even parity
  - Scaling, Inversion
  Odd parity
  - Symmetries
  P: Spatial Parity under spatial inversion (defined differently in 2D)
  T: Temporal Parity under time reversal
  C: Charge Parity under charge conjugation
- Theorems
  - Helmholtz Thm
  If the divergence and the curl of a vector function $\mathbf{F}(\mathbf{r})$ are specified, and if they both go to zero faster than $1/r^2$ as $r \to \infty$, and if $\mathbf{F}(\mathbf{r})$ goes to zero as $r \to \infty$, then $\mathbf{F}$ is given uniquely by

$$\mathbf{F} = -\boldsymbol{\nabla} U + \boldsymbol{\nabla} \times \mathbf{W}$$

  where

$$U(\mathbf{r}) \equiv \frac{1}{4\pi} \int \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{r}') \, \frac{1}{R} \, \mathrm{d}v' \qquad \text{and} \qquad \mathbf{W}(\mathbf{r}) \equiv \frac{1}{4\pi} \int \boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}') \, \frac{1}{R} \, \mathrm{d}v'$$

  - Corollary to Helmholtz Thm
  Any (differentiable) vector function $\mathbf{F}(\mathbf{r})$ that goes to zero faster than $1/r$ as $r \to \infty$ can be expressed as the gradient of a scalar plus the curl of a vector

$$\mathbf{F}(\mathbf{r}) = \boldsymbol{\nabla}\left(-\frac{1}{4\pi} \int \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{r}') \, \frac{1}{R} \, \mathrm{d}v'\right) + \boldsymbol{\nabla} \times \left(\frac{1}{4\pi} \int \boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}') \, \frac{1}{R} \, \mathrm{d}v'\right)$$

– (Griffiths pg 557, footnote 2)
As a matter of fact, any differentiable vector function *whatever* (regarless of its behavior at infinity) can be written as a gradient plus a curl, but this more general result does not follow directly from the Helmholtz theorem, nor does the corollary above supply the explicit construction, since the integrals, in general, diverge.

– Dirac delta representations and relations

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{n}}}{R^2}\right) = 4\pi\,\delta(\mathbf{R}) \qquad \boldsymbol{\nabla}'\cdot\left(\frac{\hat{\mathbf{n}}}{R^2}\right) = -4\pi\,\delta(\mathbf{R})$$

$$\boldsymbol{\nabla}'\cdot\left(\frac{\hat{\mathbf{n}}}{R^2}\right) = \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{n}}}{R^2}\right) \text{``}\left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\mathbf{r}'}\right)\text{''} = 4\pi\,\delta(\mathbf{R})(-1)$$

$$\boldsymbol{\nabla}\frac{1}{R} = -\frac{\hat{\mathbf{n}}}{R^2} \qquad \boldsymbol{\nabla}'\frac{1}{R} = \frac{\hat{\mathbf{n}}}{R^2}$$

$$\nabla^2\left(\frac{1}{R}\right) = -4\pi\,\delta(\mathbf{R}) \qquad \nabla'^2\left(\frac{1}{R}\right) = -4\pi\,\delta(\mathbf{R})$$

$$\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-y)z}\,\mathrm{d}z$$

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{\partial f}{\partial x}\right)_{x=x_i}\right|}$$

where x_i are the roots of $f(x)$

– Spherical Harmonic Addition Theorem:
When Θ is the angle between $\mathbf{r}_1 = \langle r_1, \theta_1, \phi_1\rangle_\mathrm{s}$ and $\mathbf{r}_2 = \langle r_2, \theta_2, \phi_2\rangle_\mathrm{s}$ such that

$$\cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2),$$

the Legendre polynomial of argument $\cos\Theta$ is given by

$$\begin{aligned} P_l(\cos\Theta) &= \frac{4\pi}{2l+1}\sum_{m=-l}^{l}(-1)^m Y_{lm}(\theta_1,\phi_1)\,Y_{l,-m}(\theta_2,\phi_2) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta_1,\phi_1)\,Y_{lm}^*(\theta_2,\phi_2) \\ &= P_l(\cos\theta_1)\,P_l(\cos\theta_2) + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_1)\,P_l^m(\cos\theta_2)\cos[m(\phi_1-\phi_2)]. \end{aligned}$$

– Lesser-Known Vector Calculus Thms
Green's reciprocity thm (Griffiths pg 157)

5 Terms and Quantities

See eponymous terms below in §27.

5.1 Physical Terms

- **Macroscopic** and **Microscopic**
  (all variables will be macroscopic unless stated otherwise; look at the entries for charge and current to see the only thorough treatments of variables with respect to both domains)

- **Particle**
  In classical contexts, this is intuitive and refers to tiny bits of matter (with mass) that we're familiar with. In quantum mechanical contexts, particle-wave duality exists and the term particle may include massless (or nearly massless) particles such as the photon. (These notes will stay mostly within classical context.)

- ***Q***: **Electric Charge** ("Charge")
  (positive, negative), Standard charge/current (positive/the amount of positive charge flowing in the sense of the "current arrow" on a circuit schematic), free and bound charge/current, (macroscopic and microscopic current)
  - q: a point charge (highly localized charge)
  - λ: macroscopic charge length-density (linear charge density)
  - Σ: macroscopic charge area-density (surface charge density)
    Symbol chosen so as not to be confused with conductivity σ
  - ρ: macroscopic charge volume-density

    Griffith's notation:
    $\rho = \rho^{\mathrm{f}} + \rho^{\mathrm{b}} \qquad = \rho^{\mathrm{f}} - \boldsymbol{\nabla}\cdot\mathbf{P}$
    - ρ^{b} = <u>b</u>ound charge density, which is defined to be due to polarization. (pg 170) The word "bound" reminds us that this charge cannot be removed; in a dielectric every electron is attached to a specific atom or molecule. $\rho^{\mathrm{b}} = -\boldsymbol{\nabla}\cdot\mathbf{P}$
    - ρ^{f} = <u>f</u>ree charge density. (pg 175-6) For want of a beter term, we call all other charge "free" – any charge that is not a result of polarization. The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever. (The auxiliary) Gauss's law, in the context of dielectrics, makes reference only to free charges, and free charge is the stuff we control. [or, at least the stuff that is possible to be controlled directly] (pg 328) . . . inside matter there will be accumulations of "bound" charge and current over which you exert no direct control.

    More accurate and complete notation:
    $\rho = \rho^{\mathrm{N}} + \rho^{\mathrm{P}}$
    - ρ^{P} = bound electric multi<u>p</u>olarization charge density
    - ρ^{N} = <u>n</u>ot-bound-polarization charge density $= \rho^{\mathrm{F}} + \rho^{\mathrm{B}} + \rho^{\mathrm{C}}$

    In more detail:
    $\rho = \rho^{\mathrm{Macro}} = \rho^{\mathrm{F}} + \rho^{\mathrm{B}} + \rho^{\mathrm{P}} + \rho^{\mathrm{C}}$
    - ρ^{Macro} = macroscopic (in space and time) charge density, which ignores small scale variations;
    - ρ^{F} = macroscopic "<u>f</u>ree" charge density that is due to uncomplicated free charge distributions (as opposed to complicated distributions that cause local, microscopic multipolarizations; see ρ^{C} below), where "free" means the charge carriers' motion is unimpeded in at least one direction (such as along a conductor);
    - ρ^{B} = macroscopic "<u>b</u>ound" charge density due to charges being "glued" to something or attached by static cling (make this more rigorous);

* ρ^{P} = macroscopic "bound" charge density that is a local excess charge density due to microscopic electric multipolarizations, usually just (di)polarization, where "bound" means that the charge carriers are only free to move microscopically (usually in a small, approximately parabolic potential well); and
* ρ^{C} = macroscopic "free" charge density that is due to complicated free charge distributions that have microscopic multipolarizations.[2]

Usually the microscopic charge densities are too complicated to want write out explicitly.
$\rho^{\text{micro}} = \rho^{\text{mF}} + \rho^{\text{mB}} + \rho^{\text{mP}} + \rho^{\text{mM}} + \rho^{\text{mC}}$
Most of the components are just the microscopic versions of the charge densities defined above (with all the atomic, particulate detail that one would like to ignore on a macroscopic scale), but there's one additional term:

* ρ^{mM} = microscopic (in space and time) "bound" charge density that causes microscopic magnetic multipolarizations (usually just (di)polarization) that are due to electrical charge rotation (rather than quantum-mechanical spin)

In general, $\rho^{\text{micro}}(\mathbf{x}) = \sum_i q_i\,\delta\big(\mathbf{x} - \mathbf{x}_i(t)\big)$ (classical, not quantum-mechanical expression)

- Q: total charge

- ***I*: Effective Electric-Charge Current** ("Current")
effective, meaning it includes electric current and spin magnetic effects

  - **K**: (Effective) Surface Current Density (effective electric-charge surface-current density)
  (can spin create an effective surface current density?)
  - **J**: Effective Current Density (effective electric-charge current density)
  Griffith's notation:
  $\mathbf{J} = \mathbf{J}^{\mathrm{f}} + \mathbf{J}^{\mathrm{b}} + \mathbf{J}^{\mathrm{p}} \qquad = \mathbf{J}^{\mathrm{f}} + \boldsymbol{\nabla}\times\mathbf{M} + \partial_t\mathbf{P}$

    * $\mathbf{J}^{\mathrm{b}}$ = bound current density. (pg 329) This involves the spin and orbital motion of electrons. (With regard to the electric continuity equation, an inhomogeneous magnetization does not lead to any accumulation of charge or current as an inhomogeneous polarization does.) $\mathbf{J}^{\mathrm{b}} = \boldsymbol{\nabla}\times\mathbf{M}$
    * $\mathbf{J}^{\mathrm{p}}$ = polarization current density. [but this seems to be a "bound" current density as well since it is not directly under our control] (pg 328-9) This involves the [linear motion] of bound charge, but has nothing whatever to do with the bound current $\mathbf{J}^{\mathrm{b}}$. $\mathbf{J}^{\mathrm{p}} = \partial_t\mathbf{P}$
    * $\mathbf{J}^{\mathrm{f}}$ = free current density. All other current densities. [The current densities that we have direct control over? I would say not necessarily.]
    * $\mathbf{J}^{\mathrm{d}}$ = displacement current density. $\mathbf{J}^{\mathrm{d}} = \partial_t\mathbf{D}$

  More accurate and complete notation:
  $\mathbf{J} = \mathbf{J}^{\mathrm{N}} + \mathbf{J}^{\mathrm{P}} + \mathbf{J}^{\mathrm{M}}$

    * $\mathbf{J}^{\mathrm{P}}$ = bound electric multipolarization current density
    * $\mathbf{J}^{\mathrm{M}}$ = bound magnetic multipolarization current density $= \mathbf{J}^{\mathrm{eM}} + \mathbf{J}^{\mathrm{sM}}$
    * $\mathbf{J}^{\mathrm{N}}$ = not-bound-multipolarization current density $= \mathbf{J}^{\mathrm{F}} + \mathbf{J}^{\mathrm{B}} + \mathbf{J}^{\mathrm{C}}$

  In more detail:
  $\mathbf{J} = \mathbf{J}^{\text{Macro}} = \mathbf{J}^{\mathrm{F}} + \mathbf{J}^{\mathrm{B}} + \mathbf{J}^{\mathrm{P}} + \mathbf{J}^{\mathrm{eM}} + \mathbf{J}^{\mathrm{sM}} + \mathbf{J}^{\mathrm{C}}$

    * $\mathbf{J}^{\text{Macro}}$ = macroscopic (in space and time) current density;

[2] I've never seen this kind of charge distribution considered.

* $\mathbf{J}^{\rm F}$ = macroscopic "free" current density that is due to uncomplicated free current distributions (as opposed to complicated distributions that cause local, microscopic multipolarizations; see $\mathbf{J}^{\rm C}$ below);
* $\mathbf{J}^{\rm B}$ = macroscopic "bound" current density due to charges being "glued" in to a moving object;
* $\mathbf{J}^{\rm P}$ (di=polarization part = $\partial_t \mathbf{P}$, sometimes or always?)
  = macroscopic "bound" current density that is a local excess current due to microscopic electric multipolarizations, usually just (di)polarization;
* $\mathbf{J}^{\rm M} = \mathbf{J}^{\rm eM} + \mathbf{J}^{\rm sM}$ (di-polarization part = $\boldsymbol{\nabla} \times \mathbf{M}$, sometimes or always?)
  = macroscopic "bound" effective current density that is a local excess current (or "current") due to microscopic magnetic multipolarizations, usually just (di)polarization;
  - $\mathbf{J}^{\rm eM}$ = macroscopic "bound" current density that is a local excess current due to microscopic magnetic multipolarizations due to electric charge, usually just (di)polarization;
  - $\mathbf{J}^{\rm sM}$ = macroscopic "bound" effective current density that is a local excess "current" due to microscopic magnetic (di)polarization due to quantum-mechanical spin (could there be higher multipoles?); and
* $\mathbf{J}^{\rm C}$ = macroscopic "free" current density that is due to free current distributions that have microscopic multipolarizations.[3]

So the true electric charge current density, as we've defined it, is $\mathbf{J} - \mathbf{J}^{\rm sM}$
In general, $\mathbf{J}^{\rm micro}(\mathbf{x}) = \mathbf{v}\rho^{\rm micro} = \sum_i q_i \mathbf{v}\, \delta\big(\mathbf{x} - \mathbf{x}_i(t)\big)$ (classical, not quantum-mechanical expression)

- **E: Electric Field**
  (concept that helps explain pattern of forces that can seem to remain fixed and invisible – not necessary with particle explanation of forces)
  $\mathbf{E} \equiv \mathbf{F}_q / q$

- **B: Magnetic Field** (Magnetic Flux Density, Magnetic Induction)
  a vector field $\mathbf{B}$ whose direction is defined by the direction that an infinitesimal magnetic dipole (that is able to lose energy) becomes aligned and whose magnitude is defined by the mechanical torque $\boldsymbol{\tau}$ exerted on the magnetic dipole: $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$, where $\boldsymbol{\mu}$ is the magnetic dipole moment of the magnetic dipole.
  – Note that damping of the dipole's oscillation is required for alignment to occur.
  – Here, infinitesimal means that the magnetic dipole must be "weak" enough to not perturb the existing field, small enough to reveal the local, small-scale behavior of the field, and, in an ideal sense, one should take the limit as the dipole shrinks to zero size and zero moment.

- **P: Polarization**
  or Electric Polarization (versus Mag. Polarization, or Magnetization), Polarization Density, possibly Elec. "Multipolarization"

  The electric dipole moment density, or generalized to electric multipole moment densities.
  $\mathbf{P} = \mathbf{P}^{\rm f} + \mathbf{P}^{\rm b} = \mathbf{P}^{\rm Macro} + \mathbf{P}^{\rm micro}$
  "Macro" implies a relatively smooth macroscopic function, which could include free and bound charges
  "micro" implies inclusion of complicated microscopic distributions (higher multipoles for bound charges and possibly free charges that have some complicated behavior)
  $\mathbf{P} = \mathbf{P}^{\rm coord\ ind} + \mathbf{P}^{\rm coord\ dep}$ (there are many ways to do this)

[3] I've never seen this kind of current distribution considered.

- $\mathbf{p}$: **Electric Dipole Moment**
  The electric dipole moment of the matter in a region V with charge ρ and polarization $\mathbf{P}$ is
  $\mathbf{p} \equiv \int_V \mathbf{x}\,\rho(\mathbf{x})\,\mathrm{d}v$
  $\mathbf{p}^{\mathrm{p}} \equiv \int_V \mathbf{P}\,\mathrm{d}v = \frac{1}{2}\int_V \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{P})\,\mathrm{d}v - \frac{1}{2}\oint_S \mathrm{d}\mathbf{a}\times(\mathbf{x}\times\mathbf{P}) - \frac{1}{2}\int_V(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{P}\,\mathrm{d}v$
  If V is all space and $\mathbf{P}=\mathbf{0}$ at infinity (or possibly just that $\mathbf{P}=\mathbf{0}$ on S, the boundary of V), then
  $\mathbf{p}^{\mathrm{p}} = -\int_V \mathbf{x}\,\boldsymbol{\nabla}\cdot\mathbf{P}^{\mathrm{b}}\,\mathrm{d}v$
  Thus
  $\rho^{\mathrm{p}} \equiv -\boldsymbol{\nabla}\cdot\mathbf{P}^{\mathrm{b}}$

- $\mathbf{M}$: **Magnetization**
  The magnetic dipole moment density relative to some point (designated as the origin).
  Magnetization due to current:
  $\mathbf{M}(\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}\times\mathbf{J}(\mathbf{x})$
  Magnetization due to spin:

- $\mathbf{m}$, $\boldsymbol{\mu}$: **Magnetic Dipole Moment**
  The magnetic dipole moment of the matter in a region V with magnetization $\mathbf{M}$ is
  $\mathbf{m} \equiv \frac{1}{2}\int_V \mathbf{x}\times\mathbf{J}(\mathbf{x})\,\mathrm{d}v$
  $\mathbf{m}^{\mathrm{m}} \equiv \int_V \mathbf{M}\,\mathrm{d}v = \oint_S(\mathbf{x}\cdot\mathbf{M})\,\mathrm{d}\mathbf{a} - \int_V(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{M}\,\mathrm{d}v - \int_V \mathbf{x}\times(\boldsymbol{\nabla}\times\mathbf{M})\,\mathrm{d}v$
  If V is all space and $\mathbf{M}=\mathbf{0}$ at infinity (or possibly just that $\mathbf{M}=\mathbf{0}$ on S, the boundary of V), then
  $\mathbf{m}^{\mathrm{m}} = \frac{1}{2}\int_V \mathbf{x}\times(\boldsymbol{\nabla}\times\mathbf{M})\,\mathrm{d}v$
  Thus
  $\mathbf{J}^{\mathrm{m}} \equiv \boldsymbol{\nabla}\times\mathbf{M}$
  - For a loop of current: $\mathbf{m} = I\int \mathrm{d}\mathbf{a} = I\mathbf{a}$

- χ_{e}: **Electric Susceptibility**

- χ_{m}: **Magnetic Susceptibility**
  "…in a magnet, the magnetic susceptibility can be estimated by assuming that the energy cost of flipping an electron spin is on the order of a tenth of an eV, as we would expect from atomic physics." (Peskin [3] pg 266)

- ε: **Electric Permittivity** (Dielectric Function/Constant/Tensor)

  When $\varepsilon\in\mathbb{R}$, the phase velocity in the medium is $v = 1/\sqrt{\varepsilon\mu}$ and the refractive index $n = c/v = c\sqrt{\varepsilon\mu}$

  When $\varepsilon\in\mathbb{C}$, how does one obtain the phase velocity?

  Progressive generalization:

  - ε^{b}: the "normal" real permittivity that describes polarization of bound charges
    (Usually written as ε, sometimes written as ε_{b} in Jackson)
  - ε^{c}: the real permittivity that includes polarization and conductivity (does this exist?) $\varepsilon^{\mathrm{c}} = ?$
  - $\underset{\sim}{\varepsilon}^{\mathrm{d}}$: the complex permittivity that includes polarization and damping forces (dissipation/absorption)
  - $\underset{\sim}{\varepsilon}$: the complex permittivity that includes the effects of polarization, damping forces for oscillation, conductivity, and relaxation time for current

  refractive index $\underset{\sim}{n}(\omega) = \sqrt{\mu_{\mathrm{r}}\,\underset{\sim}{\varepsilon}_{\mathrm{r}}(\omega)} = n_{\mathrm{n}} + i\kappa$ (n_{n}: "normal" index; κ: extinction coeff)
  (angular) wave vector $\underset{\sim}{\mathbf{k}}$, $\underset{\sim}{\mathbf{k}}^2 = k^2\,\hat{\underset{\sim}{\mathbf{k}}}\cdot\hat{\underset{\sim}{\mathbf{k}}} = \mu\underset{\sim}{\varepsilon}\,\omega^2 = \underset{\sim}{n}^2\,(\omega/c)^2$ (a complex vector... ouch!)
  (angular) wavenumber $\underset{\sim}{k} = \sqrt{\mu\underset{\sim}{\varepsilon}}\,\omega = \underset{\sim}{n}\,(\omega/c) = \beta + i\alpha/2$ (angular spatial freq) (β: ?; α: attenuation constant or absorption coeff)

A simple model for the dielectric function $\underset{\sim}{\varepsilon}(\omega)$ (appropriate for low density substances, small oscillations, and neglecting magnetic effects) which includes conductivity:

$$\underset{\sim}{\varepsilon}(\omega) = \varepsilon_0 + \frac{Ne^2}{m} \sum_j \frac{f_j}{{\omega_j}^2 - \omega^2 - i\omega\gamma_j}$$

- $\boldsymbol{\mu}$: **Magnetic Permeability**
- **D**: **Auxiliary Electric Field** (Electric Displacement, Effective Electric Field in a Dielectric)
  $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$
  Should the **D**-field be considered the "total electric field" while the **E**-field be considered the "applied electric field" (or "the electric field from non-local sources")?
- **H**: **Auxiliary Magnetic Field** (Magnetic Field (Strength), Effective Magnetic Field in a Dielectric)
  $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$
  Sometimes the **B**-field is called "the total magnetic field" while the **H**-field is called "the applied magnetic field", but I think a more accurate phrase would be "the magnetic field from non-local sources".
- $\boldsymbol{\Phi}$: **Electrodynamic Potential** (Scalar Potential, "Potential")
  - $\boldsymbol{V}$ **Voltage** (anything with units of volts?)
  - $\boldsymbol{V}(\mathbf{r})$ in electrostatics: electric potential energy $\mathcal{V} = qV$
  - $\boldsymbol{V}(\mathbf{r}), \boldsymbol{t}$ or $\boldsymbol{\Phi}(\mathbf{r}, \boldsymbol{t})$ in electrodynamics: $\mathcal{U} = q(\Phi - \mathbf{A} \cdot \mathbf{v})$

  This is a generalized electric potential. In other words, it is not related to a conservative potential and is not properly a potential in that common sense, but it can be used to calculate the electric field (a force per charge) as a generalized potential can be used to calculate forces.
- $\boldsymbol{\mathcal{E}}$: **Electromotive Force** (EMF, emf, Electromotance)
  Not really a force, more properly "electromotive work" or "electromotive generalized electric potential"(?) since it is time-dependent

  $$\mathcal{E} \equiv \int \mathbf{E} \cdot \mathrm{d}\mathbf{s} \qquad \text{or} \qquad \mathcal{E} \equiv \int \mathbf{f} \cdot \mathrm{d}\mathbf{s} \quad ?$$

  where **f** is the force per charge (e.g., $\mathbf{f}_B = \mathbf{F}_B/q = \mathbf{v} \times \mathbf{B}$)
  - seat or source of emf
  - induced emf (electromagnetic induction)
  - motional emf
  - back or counter emf
- **A**: **Magnetic Potential** (Vector Potential)
- $\boldsymbol{W}$: **Work**, Conservative fields/forces, Scalar potential
- Capacitance, Energy density of fields, Inductance
- **S**: **EM Energy Flux Density** (EM power area-density)

  $$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

- $\mathbf{g}$: **EM Momentum Density**
  Generally accepted expression for a medium at rest:

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \mu_0\varepsilon_0\mathbf{E}\times\mathbf{H}$$

  Using the macroscopic Maxwell eqns leads to an apparent electromagnetic momentum density $\mathbf{g} = \mathbf{D}\times\mathbf{B}$ (Minkowski, 1908), which yields, for a plane wave, the "pseudomomentum" of the wave vector ($k = n\omega/c$ or $\hbar k = n\hbar\omega/c$ for a photon)

- $\mathbf{v}_\mathrm{p}$: **Phase Velocity**
  (not to be confused with the Hamiltonian phase velocity)
  $\mathbf{v}_\mathrm{p} = \omega/k$ (complex form?)

- $\mathbf{v}_\mathrm{g}$: **Group Velocity**
  $\mathbf{v}_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k$ (complex form?)

- **Special Relativity Terms**

  See the *Notes on Special and General Relativity* for terms such as A_μ, the four-potential, and $F^{\mu\nu}$, the EM field-strength tensor. (Some definitions will be repeated within these notes, as the need arises.)

5.2 Mathematical Terms

Are these really "mathematical terms"?

- Thread
- Circulation
- Vector area (Griffiths pg 57)
- Irrotational, Incompressible (Flow), Solonoidal (is this a good term?)
- Generic Scalar ϕ Transport Equation

$$\frac{\partial\rho\phi}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\phi) = \boldsymbol{\nabla}\cdot(\Gamma\boldsymbol{\nabla}\phi) + S_\phi$$

  1. Transient term: $\frac{\partial\rho\phi}{\partial t}$
  2. Convection term: $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}\phi)$
  3. Diffusion term: $\boldsymbol{\nabla}\cdot(\Gamma\boldsymbol{\nabla}\phi)$
  4. Source/Sink term: S_ϕ (could include radiation)

  For example, in transport phenomena (diffusion, convection, and radiation), there is heat transfer, mass transfer, and momentum transfer (in fluid or field(?) dynamics)

  – Local Derivative
  – Convective Derivative (Individual, Material, Particle, Substantial Derivative)
  – Convective Term (a.k.a. Advective Term)
  – Continuity Equation

5.3 Ambiguous Terms

- Flux
- Intensity
- Terms from radiometry, etc.
- "bound"
- "free"

6 Units and Constants

Symbol	Quantity	Units	Notes
$\mathbf{E}$	Electric field	(V/m)	$[\mathbf{E}] = [\mathbf{B}]$(m/s)
$\mathbf{H}$	Auxiliary magnetic field (magnetic field strength)	(A/m)	$[\mathbf{H}] = [\mathbf{D}]$(m/s)
$\mathbf{M}$	Magnetization (magnetic dipole moment density)	(A/m)	$[\mathbf{M}] = [\mathbf{m}]/\mathrm{m}^3$
$\mathbf{P}$	Polarization (electric dipole moment density)	($\mathrm{C/m^2}$)	$[\mathbf{P}] = [\mathbf{p}]/\mathrm{m}^3$
$\mathbf{D}$	Auxiliary electric field (elec. displacement, flux density?)	($\mathrm{C/m^2}$)	A = C/s
$\mathbf{B}$	Magnetic flux density (magnetic induction)	($\mathrm{Wb/m^2}$)	V = Wb/s
ρ	Charge density (electric monopole mom. dens.)	($\mathrm{C/m^3}$)	
$\mathbf{J}$	Effective current density	($\mathrm{A/m^2}$)	$[\mathbf{J}] = [\rho]$(m/s)
Φ	Electric scalar potential	(V)	V = J/C
$\mathbf{A}$	Magnetic vector potential	(Wb/m)	Wb = J/A
$\mathbf{m}$, $\boldsymbol{\mu}$	Magnetic dipole moment	($\mathrm{A \cdot m^2}$)	
$\mathbf{p}$	Electric dipole moment	($\mathrm{C \cdot m}$)	

free space permittivity: $\varepsilon_0 = (\frac{1}{4\pi c^2} \times 10^7\ \mathrm{F \cdot m/s^2}) = (8.85 \times 10^{-12}\ \mathrm{F/m})$ $\quad (\mathrm{F/m} = \frac{\mathrm{C^2 s^2}}{\mathrm{kg \cdot m^3}})$

free space permeability: $\mu_0 = (4\pi \times 10^{-7}\ \mathrm{H/m}) = (1.26 \times 10^{-6}\ \mathrm{H/m})$ $\quad (\mathrm{H/m} = \frac{\mathrm{kg \cdot m}}{\mathrm{C^2}})$

free space impedance: $Z_0 = \sqrt{\mu_0/\varepsilon_0} = \mu_0 c = 1/\varepsilon_0 c = (376.7\ \Omega)$

(What does this "mean"? Is it conceptually kind of like inertia/spring constant/damping(no!)?)

permittivity and electric susceptance: $\varepsilon = (1 + \chi_\mathrm{e})\varepsilon_0$ (linear media)

permeability and magnetic susceptance: $\mu = (1 + \chi_\mathrm{m})\mu_0$ (linear media)

electrostatic constant: $K_\mathrm{e} = \frac{1}{4\pi\varepsilon_0} = (8.99 \times 10^9\ \mathrm{N}\frac{\mathrm{m^2}}{\mathrm{C^2}})$ $\quad$ ($= k_\mathrm{C}$ Coulomb force constant)

magnetostatic constant: $K_\mathrm{m} = \frac{\mu_0}{4\pi} = (10^{-7}\ \mathrm{N/A^2})$

$[K_\mathrm{m}] = [\mu_0] = [\frac{\int \boldsymbol{\nabla}\times\mathbf{B} \cdot d\mathbf{a}}{I}] = (\frac{\mathrm{Tm}\cancel{^2}}{\mathrm{A}\cancel{\mathrm{m}}}) = [\frac{\int \mathbf{B} \cdot d\mathbf{s}}{I}] = [\frac{B_\mathrm{solenoid}}{K}] = (\frac{\mathrm{Tm}}{\mathrm{A}}) = (\frac{\mathrm{N}\cancel{\mathrm{m}}}{\mathrm{A^2}\cancel{\mathrm{m}}}) = [?] = (\frac{\mathrm{N}}{\mathrm{A^2}}) = (\frac{\mathrm{Nsm}}{\mathrm{ACm}}) = (\frac{\mathrm{Vs}}{\mathrm{Am}}) = (\Omega\frac{\mathrm{s}}{\mathrm{m}})$

$(\mathrm{T}) = [\mathbf{B}] = [\frac{\tau}{Ia}] = (\frac{\mathrm{N}\cancel{\mathrm{m}}}{\mathrm{Am}\cancel{^2}}) = [\frac{F}{qv}] = (\frac{\mathrm{N}}{\mathrm{Cm/s}}) = [\frac{F}{I\ell}] = (\frac{\mathrm{N}}{\mathrm{Am}})$

$= (\frac{\mathrm{kg}}{\mathrm{As^2}}) = (\frac{\mathrm{kg}}{\mathrm{Cs}})$

$[\frac{1}{\mu_0}\mathbf{B}^2] = (\frac{\mathrm{Js}}{\mathrm{Cm^2}})^2/(\frac{\mathrm{kg \cdot m}}{\mathrm{C^2}}) = (\mathrm{J}\frac{\mathrm{Js^2}}{\mathrm{C^2m^4}})(\frac{\mathrm{C^2}}{\mathrm{kg \cdot m}}) = (\mathrm{J}\frac{\mathrm{kg}}{\mathrm{C^2m^2}})(\frac{\mathrm{C^2}}{\mathrm{kg \cdot m}}) = (\mathrm{J/m^3})$

$[\Phi] = \mathrm{V} = \mathrm{J/C} = \mathrm{W/A}$	$[\mathbf{A}] = \mathrm{Wb/m} = \mathrm{N/A} = \mathrm{J/Am}$
$\mathrm{F} = \mathrm{C/V} = \mathrm{C^2/J}$	$\mathrm{H} = \mathrm{Wb/A} = \mathrm{J/A^2}$

7 Theoretical Summary

7.1 Principles

- Relativity
- Superposition (Linearity): Construction using elements and points
  - If it were $F_q \propto q^2$, then $F_{q_1+q_2} \propto (q_1+q_2)^2 \neq {q_1}^2 + {q_2}^2 \propto F_{q_1} + F_{q_2}$ (Griffiths pg 58)
  - Linearity of Maxwell's equations.
  - More advanced: Photons don't have charge. (Yet they do interact with each other…?)
  - What about non-linear media?
- Perturbation: source charges, test charges; source, test currents.
  - See questions above.

7.2 Instantaneous Equations

(Local Space-Time Equations?)

Figure out what is the nicest convention. Should ρ and J represent the free charge and current or the total charge and current. How inclusive are D and H?

What are all of the possible sources? (Charge: free charges in space (small in mass), mobile conduction charges (small in mass), bound charges of a conductor (massive), bound charges in a dielectric, bound charges on an insulator, charges in electrostatic equilibrium on a moving conductor,...)

PDE theory: Discuss the number of equations in the theory and the boundary conditions necessary to yield a particular solution. (Electromagnetic field equations versus Potential field equations, plus [generalized?] Lorentz force equation)

- Newton force law (3), Maxwell Eqns (8, implying continuity eqn), Lorentz force law (3), BCs

Maxwell Eqns	**Differential Form**	**Integral Form**
1. Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$	$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0}Q_{\text{encl}}$
2. Magnetic Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{B} = 0$	$\oint_S \mathbf{B}\cdot\mathrm{d}\mathbf{a} = 0$
3. Faraday's Law	$\boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0}$	$\oint_C \mathbf{E}\cdot\mathrm{d}\mathbf{s} + \partial_t\Phi_B = 0$
4. Ampère-Maxwell Law	$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0\mathbf{J}$	$\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{s} - \frac{1}{c^2}\partial_t\Phi_E = \mu_0 I_{\text{enc}}$

encl = enclosed by the surface S, enc = encircled by the path or contour C

I think that the integral forms may only valid locally in spacetime (due to retardation of fields). (These are the three dimensional integrals, but what about the four-vector / EM tensor integrals?)

Gauss's law and the Ampère-Maxwell law in more detail:

Expanded Differential Form	**Expanded Integral Form**
$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\left[\rho^{\text{f}} - \boldsymbol{\nabla}\cdot\mathbf{P}\right]$	$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0}\left[Q^{\text{f}}_{\text{encl}} - \oint_S \mathbf{P}\cdot\mathrm{d}\mathbf{a}\right]$
$= \frac{1}{\varepsilon_0}\left[\rho^{\text{F}} + \rho^{\text{B}} + \rho^{\text{P}} + \rho^{\text{C}}\right]$	$= \frac{1}{\varepsilon_0}\left[Q^{\text{F}}_{\text{encl}} + Q^{\text{B}}_{\text{encl}} + Q^{\text{P}}_{\text{encl}} + Q^{\text{C}}_{\text{encl}}\right]$
$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0\left[\mathbf{J}^{\text{f}} + \partial_t\mathbf{P} + \boldsymbol{\nabla}\times\mathbf{M}\right]$	$\oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{s} - \frac{1}{c^2}\partial_t\Phi_E = \mu_0\left[I^{\text{f}}_{\text{enc}} + \partial_t\Phi_P + \oint_C \mathbf{M}\cdot\mathrm{d}\mathbf{s}\right]$
$= \mu_0\left[\mathbf{J}^{\text{F}} + \mathbf{J}^{\text{B}} + \mathbf{J}^{\text{P}} + \mathbf{J}^{\text{eM}} + \mathbf{J}^{\text{sM}} + \mathbf{J}^{\text{C}}\right]$	$= \mu_0\left[I^{\text{F}}_{\text{enc}} + I^{\text{B}}_{\text{enc}} + I^{\text{P}}_{\text{enc}} + I^{\text{eM}}_{\text{enc}} + I^{\text{sM}}_{\text{enc}} + I^{\text{C}}_{\text{enc}}\right]$

"Auxiliary Maxwell Eqns"	**Differential Form**	**Integral Form**
1. Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho^{\text{f}}$	$\oint_S \mathbf{D}\cdot\mathrm{d}\mathbf{a} = Q^{\text{f}}_{\text{encl}}$
2. Magnetic Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{B} = 0$	$\oint_S \mathbf{B}\cdot\mathrm{d}\mathbf{a} = 0$
3. Faraday's Law	$\boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0}$	$\oint_C \mathbf{E}\cdot\mathrm{d}\mathbf{s} + \partial_t\Phi_B = 0$
4. Ampère-Maxwell Law	$\boldsymbol{\nabla}\times\mathbf{H} - \partial_t\mathbf{D} = \mathbf{J}^{\text{f}}$	$\oint_C \mathbf{H}\cdot\mathrm{d}\mathbf{s} - \partial_t\Phi_D = I^{\text{f}}_{\text{enc}}$

f = "free", but that's not really the whole story, see below…

In more detail (generalizing **P** and **M**, and thus **D** and **H**, to include multipolarizations):

"Auxiliary Maxwell Eqns"	**Differential Form**	**Integral Form**
1. Gauss's Law	$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho^{\text{F}} + \rho^{\text{B}} + \rho^{\text{C}}$	$\oint_S \mathbf{D}\cdot\mathrm{d}\mathbf{a} = Q^{\text{F}}_{\text{encl}} + Q^{\text{B}}_{\text{encl}} + Q^{\text{C}}_{\text{encl}}$
4. Ampère-Maxwell Law	$\boldsymbol{\nabla}\times\mathbf{H} - \partial_t\mathbf{D} = \mathbf{J}^{\text{F}} + \mathbf{J}^{\text{B}} + \mathbf{J}^{\text{C}}$	$\oint_C \mathbf{H}\cdot\mathrm{d}\mathbf{s} - \partial_t\Phi_D = I^{\text{F}}_{\text{enc}} + I^{\text{B}}_{\text{enc}} + I^{\text{C}}_{\text{enc}}$

"Remaining Maxwell Eqns" (implied by constitutive relations)	
$\boldsymbol{\nabla}\cdot\mathbf{H} = -\boldsymbol{\nabla}\cdot\mathbf{M}$	$\oint_S \mathbf{H}\cdot\mathrm{d}\mathbf{a} = -\oint_S \mathbf{M}\cdot\mathrm{d}\mathbf{a}$
$\boldsymbol{\nabla}\times\mathbf{D} + \frac{1}{c^2}\partial_t\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{P} - \frac{1}{c^2}\partial_t\mathbf{M}$	$\oint_C \mathbf{D}\cdot\mathrm{d}\mathbf{s} + \frac{1}{c^2}\partial_t\Phi_H = \oint_C \mathbf{P}\cdot\mathrm{d}\mathbf{s} - \frac{1}{c^2}\partial_t\Phi_M$

Constitutive Relations	**Simple Forms**	**Linear Media**		
$\mathbf{D} = \mathbf{D}[\mathbf{E},\mathbf{B}]$	$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{P} = \varepsilon_0\chi_{\text{e}}\mathbf{E} \;\Rightarrow$	$\mathbf{D} = \varepsilon\mathbf{E}$	$\varepsilon \equiv (1+\chi_{\text{e}})\varepsilon_0$
$\mathbf{H} = \mathbf{H}[\mathbf{E},\mathbf{B}]$	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	$\mathbf{M} = \chi_{\text{m}}\mathbf{H} \;\Rightarrow$	$\mathbf{B} = \mu\mathbf{H}$	$\mu \equiv (1+\chi_{\text{m}})\mu_0$

The square brackets signify that the connections are not necessarily simple and may depend on past history (hysteresis), may be non-linear, etc.

More complicated expressions are
$D_\alpha = \sum_\beta \varepsilon_{\alpha\beta} E_\beta$, where $\varepsilon_{\alpha\beta}$ is the electric permittivity or dielectric tensor, and
$H_\alpha = \sum_\beta \mu'_{\alpha\beta} B_\beta$, where $\mu'_{\alpha\beta}$ is the inverse magnetic permeability tensor.
(See Griffiths pg 545 for a problem on Minkowski's relativistic constitutive relations.)

Manifestly (Special-)Relativistically Invariant

Maxwell Eqns	Differential Form	Integral Form	Comment
1. Homogeneous Law	$\partial_\nu \widetilde{F}^{\mu\nu} = 0^\mu$		True identically ("gauge condition"?)
2. Inhomogeneous Law	$\partial_\nu F^{\mu\nu} = J^\mu$		

$\widetilde{F}^{\mu\nu}$ is the dual of the field-strength tensor $F^{\mu\nu}$ ($\widetilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} g_{\rho\alpha} g_{\sigma\beta} F^{\alpha\beta}$)

Lorentz Force	Generalized, Quasi-Static Lorentz Force
$\mathbf{F}_q = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$	$\mathbf{F} = \int_V (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\mathrm{d}v$ $+ \int_S (\sigma\mathbf{E} + \mathbf{K}\times\mathbf{B})\mathrm{d}a$ $+ \int_C (\lambda\mathbf{E} + \mathbf{I}\times\mathbf{B})\mathrm{d}s$ $+ \sum_i q_i(\mathbf{E} + \mathbf{v}_i\times\mathbf{B})$

σ, λ, and each q_i may be included in ρ, using Dirac deltas, yeilding only one integral.
Did I put "Quasi-Static" because these integrals are not retarded? Or is there something else (also)?

Potentials and Gauge	
$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t\mathbf{A}$	$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$
$\Phi' = \Phi - \partial_t\lambda$	$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda$

- Coulomb, transverse, radiation gauge: $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$
- Lorenz gauge: $\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\partial_t\Phi = 0$
- Weyl, Hamiltonian, temporal gauge (an incomplete gauge): $\Phi = 0$
- Multipolar, line, Poincaré gauge: $\mathbf{x}\cdot\mathbf{A} = 0$
- Fock-Schwinger, relativistic Poincaré gauge: $x^\mu A_\mu = 0$
- (R_ξ gauges? Landau, Feynman-t'Hooft, Yennie gauges?)

Potentials Eqns	Differential Form	Comment
1. Gauss's Law	$-\nabla^2\Phi - \partial_t\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\varepsilon_0}\rho^{\mathrm{f}} - \frac{1}{\varepsilon_0}\boldsymbol{\nabla}\cdot\mathbf{P}$	
2. Magnetic Gauss's Law	$\cancel{\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A})} = 0$	True identically
3. Faraday's Law	$-\cancel{\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Phi} - \cancel{\partial_t\boldsymbol{\nabla}\times\mathbf{A}} + \cancel{\partial_t\mathbf{B}} = \mathbf{0}$	True identically
4. Ampère-Maxwell Law	$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} + \frac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi + \frac{1}{c^2}{\partial_t}^2\mathbf{A} =$ $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A} + \frac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi + \frac{1}{c^2}{\partial_t}^2\mathbf{A} =$ $\frac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \partial^2\mathbf{A} = \mu_0\mathbf{J}^{\mathrm{f}} + \mu_0\boldsymbol{\nabla}\times\mathbf{M} + \mu_0\partial_t\mathbf{P}$	

Potentials Eqns	Integral Form	Comment
1. Gauss's Law		
2. Magnetic Gauss's Law		True identically
3. Faraday's Law		True identically
4. Ampère-Maxwell Law		

Potentials Eqns	Differential Form	Comment
1. Homogeneous?	(gauge?)	
2. Inhomogeneous	$\Box^2 A_\mu = \mu_0 J_\mu$	

Definition of Conductivity	Electrodynamical Case	Normally $v \ll c$
$\mathbf{J} = \sigma \mathbf{f}$	$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$\mathbf{J} = \sigma \mathbf{E}$

$\mathbf{J} = \mathbf{J}^{\mathrm{f}}$?

$\mathbf{f}$ is the (external) force per charge

Generalized Ohm's Law	Ohm's Law	Conductivity Model
$\mathbf{J} = \mathbf{J}[\mathbf{E}, \mathbf{B}]$	$\mathbf{J} = \sigma \mathbf{E}$	$\sigma = \ldots$

Should I bother with cgs, or other unit systems?

Generalized holor form of the equations...

Maxwell Eqns	Differential Form	Integral Form
1. Gauss's Law	$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$ $= \frac{1}{\varepsilon_0}\sum_{i=0}^{\infty} \mathrm{D}^{(i)}(P^i)$ $= \frac{1}{\varepsilon_0}\rho^{\mathrm{f}} - \frac{1}{\varepsilon_0}\nabla \cdot \mathbf{P} - \cdots$	$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\mathrm{encl}}$
2. Magnetic Gauss's Law	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$
3. Faraday's Law	$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}$	$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{s} + \partial_t \Phi_B = 0$
4. Ampère-Maxwell Law	$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E}$ $= \mu_0 \mathbf{J}^{\mathrm{f}} + \mu_0 \nabla \times \mathbf{M} + \mu_0 \partial_t \mathbf{P}$	$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{s} - \frac{1}{c^2}\partial_t \Phi_E$ $= \mu_0 I^{\mathrm{f}}_{\mathrm{enc}} + \mu_0 \oint_C \mathbf{M} \cdot \mathrm{d}\mathbf{s} + \mu_0 \partial_t \Phi_P$

7.3 Retarded Equations

Green Functions

(for the wave eqn?)

$$G^{(\pm)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right)$$

where $G^{(+)}$ is the *retarded Green function* and $G^{(-)}$ is the *advanced Green function*.

Or, on page 661, in relativistic notation where $(ct, \mathbf{x}) = x$,

$$G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') = D_{\mathrm{r}}(x - x')$$

Potentials

$$\begin{aligned}
\Phi(\mathbf{x}, t) &= K_{\mathrm{e}} \int \mathrm{d}^3x' \left[\rho(\mathbf{x}', t)\right]_{\mathrm{ret}} \frac{1}{R} \\
\mathbf{A}(\mathbf{x}, t) &= K_{\mathrm{m}} \int \mathrm{d}^3x' \left[\mathbf{J}(\mathbf{x}', t)\right]_{\mathrm{ret}} \frac{1}{R}
\end{aligned}$$

“Preliminary Forms”

$$\mathbf{E}(\mathbf{x},t) = K_{\rm e}\int {\rm d}^3x' \left[-\boldsymbol{\nabla}'\rho - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t'}\right]_{\rm ret}\frac{1}{R}$$
$$\mathbf{B}(\mathbf{x},t) = K_{\rm m}\int {\rm d}^3x' \left[\boldsymbol{\nabla}'\times\mathbf{J}\right]_{\rm ret}\frac{1}{R}$$

Note that

$$[\boldsymbol{\nabla}' f]_{\rm ret} \neq \boldsymbol{\nabla}'[f]_{\rm ret}$$

and

$$[\boldsymbol{\nabla}'\rho]_{\rm ret} = \boldsymbol{\nabla}'[\rho]_{\rm ret} - \left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}\boldsymbol{\nabla}'(t-R/c) = \boldsymbol{\nabla}'[\rho]_{\rm ret} - \frac{\hat{\mathbf{n}}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{\rm ret}$$
$$\left[\boldsymbol{\nabla}'\times\mathbf{J}\right]_{\rm ret} = \boldsymbol{\nabla}'\times[\mathbf{J}]_{\rm ret} + \left[\tfrac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}\times\boldsymbol{\nabla}'(t-R/c) = \boldsymbol{\nabla}'\times[\mathbf{J}]_{\rm ret} + \left[\tfrac{\partial\mathbf{J}}{\partial t'}\right]_{\rm ret}\times\frac{\hat{\mathbf{n}}}{c}$$

Jefimenko’s Generalizations

(of the Coulomb and Biot-Savart Laws) (“Liminary Forms”!) after integration by parts is performed on the first term in each expression (the gradient and curl terms), we have

$$\mathbf{E}(\mathbf{x},t) = K_{\rm e}\int {\rm d}^3x' \left\{\frac{\hat{\mathbf{n}}}{R^2}\left[\rho(\mathbf{x}',t')\right]_{\rm ret} + \frac{\hat{\mathbf{n}}}{cR}\left[\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\right]_{\rm ret} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\rm ret}\right\}$$
$$\mathbf{B}(\mathbf{x},t) = K_{\rm m}\int {\rm d}^3x' \left\{\left[\mathbf{J}(\mathbf{x}',t')\right]_{\rm ret}\times\frac{\hat{\mathbf{n}}}{R^2} + \left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{\rm ret}\times\frac{\hat{\mathbf{n}}}{cR}\right\}$$

Heaviside-Feynman Expressions

(for the fields of a point charge)

Let $\mathbf{r}_0$ be the position of the charge, $\rho(\mathbf{x}',t') = q\,\delta[\mathbf{x}'-\mathbf{r}_0(t')]$, $\mathbf{J}(\mathbf{x}',t') = q\,\mathbf{v}(t')$, and $\kappa \equiv 1 - \mathbf{v}\cdot\hat{\mathbf{n}}/c$ (a retardation factor).

Since $t' = t - \left|\mathbf{x}-\mathbf{x}'\right|/c$ (or $c\tau = R$),

$$\left[\frac{\partial f(\mathbf{x}',t')}{\partial t'}\right]_{\rm ret} = \frac{\partial}{\partial t}\left[f(\mathbf{x}',t')\right]_{\rm ret}$$

so

$$\mathbf{E}(\mathbf{x},t) = K_{\rm e}\,q\left\{\left[\frac{\hat{\mathbf{n}}}{\kappa R^2}\right]_{\rm ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{n}}}{\kappa R}\right]_{\rm ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\rm ret}\right\}$$
$$\mathbf{B}(\mathbf{x},t) = K_{\rm m}\,q\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{n}}}{\kappa R^2}\right]_{\rm ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{n}}}{\kappa R}\right]_{\rm ret}\right\}$$

But

$$\left[\frac{\partial f}{\partial t}\right]_{\rm ret} \neq \frac{\partial}{\partial t}[f]_{\rm ret}$$

since $\mathbf{x}' \to \mathbf{r}_0(t')$, so

$$\mathbf{E}(\mathbf{x},t) = K_\mathrm{e}\, q \left\{ \left[\frac{\hat{\mathbf{n}}}{R^2}\right]_\mathrm{ret} + \frac{[R]_\mathrm{ret}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_\mathrm{ret} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\hat{\mathbf{n}}]_\mathrm{ret} \right\}$$
$$\mathbf{B}(\mathbf{x},t) = K_\mathrm{m}\, q \left\{ \left[\frac{\mathbf{v}\times\hat{\mathbf{n}}}{\kappa R^2}\right]_\mathrm{ret} + \frac{1}{c[R]_\mathrm{ret}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{n}}}{\kappa}\right]_\mathrm{ret} \right\}$$

7.4 Relativistic Transformation of Fields: Gaussian Units

(A boost transformation $\mathbb{B}$, or "change of basis" on the spacetime manifold, gives the field values for an observer in the boosted reference frame) Transformed Field Strength Tensor:

$$F' = \mathbb{B}F\tilde{\mathbb{B}}$$

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta}$$

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})$$

$$\mathbf{B}' = \gamma(\mathbf{B} + \boldsymbol{\beta}\times\mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})$$

$$\begin{array}{ll} E_1' = E_1 & B_1' = B_1 \\ E_2' = \gamma(E_2 - \beta B_3) & B_2' = \gamma(B_2 + \beta E_3) \\ E_3' = \gamma(E_3 + \beta B_2) & B_3' = \gamma(B_3 - \beta E_2) \end{array}$$

(See Jackson eqn 11.152 *et al.*)

Mention how this relates to the fields of a charge in linear motion and length contraction.

7.5 Diagramatic Relations

- Relativistic and Retarded general diagrams (generalize Griffiths' -static diagrams)
- Griffiths diagram relating quantities and transformations
  - Electrostatic diagram (pg 87) (gauge? BCs?)

Electrostatic Relations

	ρ	
$\Phi = K_\mathrm{e}\int \rho\frac{1}{R}\mathrm{d}v'$ (+BCs?)		$\mathbf{E} = K_\mathrm{e}\int \rho\frac{\hat{\mathbf{n}}}{R^2}\mathrm{d}v'$
$-\nabla^2\Phi = \frac{1}{\varepsilon_0}\rho$		$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$; $\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$
	$\mathbf{E} = -\boldsymbol{\nabla}\Phi$	
Φ	$\Delta\Phi = -\int \mathbf{E}\cdot\mathrm{d}\mathbf{s}$	$\mathbf{E}$

  - Magnetostatic diagram (pg 240) (gauge? BCs?)

Magnetostatic Relations

	$\mathbf{J}$	
$\mathbf{A} = K_\mathrm{m}\int \mathbf{J}\frac{1}{R}\mathrm{d}v'$		$\mathbf{B} = K_\mathrm{m}\int \mathbf{J}\times\frac{\hat{\mathbf{n}}}{R^2}\mathrm{d}v'$
$-\nabla^2\mathbf{A} = \mu_0\mathbf{J}$		$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$; $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$
	$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$; $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$	
$\mathbf{A}$	(See Probs. 5.50 and 5.51)	$\mathbf{B}$

"the Biot-Savart law provides an inverse to the curl operation; the result is unique up to gauge transformation."

8 Postulates and Theorems

- **Mean Value Thm**: For charge-free space, the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.
- **Green's Reciprocation Thm**: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density Σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and Σ', then

$$\int_V \rho\,\Phi'\,\mathrm{d}^3x + \int_S \Sigma\,\Phi'\,\mathrm{d}a = \int_V \rho'\,\Phi\,\mathrm{d}^3x + \int_S \Sigma'\,\Phi\,\mathrm{d}a$$

- **Thomson's Thm**: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.
- **Optical Theorem**: The total cross section of a scatterer is related to the imaginary part of the forward (normalized) scattering amplitude by

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}\left[\boldsymbol{\epsilon}_0^* \cdot \mathbf{f}(\mathbf{k}=\mathbf{k}_0)\right]$$

  where $\mathbf{f}$ is the normalized scattering amplitude: $\mathbf{f}(\mathbf{k},\mathbf{k}_0) = \mathbf{F}(\mathbf{k},\mathbf{k}_0)/E_0$. The (unnormalized) vectorial scattering amplitude is $\mathbf{F}(\mathbf{k},\mathbf{k}_0)$, where $\mathbf{E}_\mathrm{s}(\mathbf{r}) \to \frac{e^{ikr}}{r}\mathbf{F}(\mathbf{k},\mathbf{k}_0)$.

  "forward" implies the wave vector $\mathbf{k} = \mathbf{k}_0$ and the polarization vector $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0$

  σ_t is sometimes called the extinction cross section in optics

9 Conservation Equations and Invariance

- Continuity Equation (Conservation of mass and charge)
- Poynting Thm: $\partial_t\, u + \boldsymbol{\nabla}\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$
  - $u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$, $\mathbf{S} = \mathbf{E}\times\mathbf{H}$
  - Integral form (via Div Thm): $\partial_t(E_\text{field} + E_\text{mech}) = -\int_S \mathrm{d}\mathbf{a}\cdot\mathbf{S}$
  - For linear dispersive media with losses: $\frac{\partial u_\text{eff}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E} - \omega_0\,\mathrm{Im}\,\varepsilon(\omega_0)\,\langle\mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\rangle - \omega_0\,\mathrm{Im}\,\mu(\omega_0)\,\langle\mathbf{H}(\mathbf{x},t)\cdot\mathbf{H}(\mathbf{x},t)\rangle$
  - For harmonic fields: $\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\,\mathrm{d}^3x + 2i\omega\int_V(\underset{\sim}{u}_\mathrm{e} - \underset{\sim}{u}_\mathrm{m})\,\mathrm{d}^3x + \oint_S \mathbf{S}\cdot\hat{\mathbf{n}}\,\mathrm{d}a = 0$
    - $\underset{\sim}{u}_\mathrm{e} = \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^*$, $\underset{\sim}{u}_\mathrm{m} = \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^*$
    - Real part $\Rightarrow$ conservation of energy for the time-averaged quantities
    - Imaginary part $\Rightarrow$ reactive or stored energy and its alternating flow
    - Impedance, Admittance, Reactance...
    - ...
- Conservation of energy and momentum: Stress tensor

- Scale invariance?
- Covariance of laws: Lorentz transformations
  The laws of electrodynamics are invariant under these transformations...

10 Fields in Space without Matter: Electromagnetic Waves

- Plane Waves
- Spherical Waves
- Other geometries in empty space?

– Wave view: "self-propagating" waves (i.e., no oscillating medium)

– Particle view: just photons flying in straight paths, or quantum-mechanically "wave-functioning" everywhere

∗ c is the (group) speed of light (in vacuo)

∗ The phase speed of light in vacuo (is/can be) much greater than c

∗ The group speed of light in media may be much lower

∗ The phase speed of light in media ...?

∗ Particles other than photons can travel faster than photons in media

∗ (Particles have phase and group velocities too)

11 Radiation and Fields of a Point Charge (and Beyond)

Due to linearity of the equations, we can analyze and build up the properties of a complicated system from the constituents of the system, most (if not all) times assuming those constituents are point sources. In many instances building-up shows up as an integral where, say, $\sum_i q_i$ becomes $\int \mathrm{d}^3x' \rho(\mathbf{x}')$. So once we find an expression for a point charge, we can simply transform it into a general expression for an arbitrary charge distribution.

Accelerating charged particles radiate.

11.1 Field Reaction

- Field Reaction versus Radiation Reaction
- Velocity and Acceleration Fields $\mathbf{E} = \mathbf{E}_v + \mathbf{E}_a$
- Energy cross term: $E^2 = {E_v}^2 + 2\mathbf{E}_v \cdot \mathbf{E}_a + {E_a}^2$
  Energy radiated versus non-radiated (is there a good word/concept for this?)
- Abraham-Lorentz force, Abraham-Lorentz-Dirac force
  Unphysical solutions: run-away charged particles, acausal acceleration (see Griffiths pg 467)
  Feynman's assertions versus other thoughts: `http://www.mathpages.com/home/kmath528/kmath528.htm`
  ⇒ Need for QFT?
- What is fundamental? What is the starting point for these various derivations? (the derivations noted in the website of the URI given above)

Radiation Reaction and Jerk!

Radiation reaction is the resistance to acceleration caused by the forces of energy transfer, i.e., the recoil force due to release of radiation. (Wikipedia: "It is part of the self-force of an electric charge which is the net force that a charged object's electromagnetic field exerts on the object itself. Another part of the self-force is an addition to the object's opposition to being accelerated (inertia) due to its electrical potential energy of self-repulsion." → "(effective) electromagnetic mass")

"The radiation reaction is proportional to the square of the object's charge times the jerk (rate of change of acceleration) which it is experiencing. It points in the direction of the jerk. So in a cyclotron where the jerk is pointing opposite to the velocity, the radiation reaction is similar to resistance in being directed opposite to the velocity of the particle."

Abraham-Lorentz force (of field reaction):

$$\begin{aligned}\mathbf{F} &= \frac{2}{3} K_{\mathrm{e}} \frac{q^2}{c^3}\, \dot{\mathbf{a}} \\ &= \frac{2}{3} K_{\mathrm{e}}\, q^2\, {\partial_0}^3 \mathbf{r}\end{aligned}$$

(see Griffiths pg 467 and following section: this is the "self-force" of the particle's fields on itself)

Abraham-Lorentz-Dirac force (of field reaction?):

$$\begin{aligned}F^{\mu}_{\mathrm{rad}} &= \frac{2}{3} K_{\mathrm{e}} \frac{q^2}{c^3} \left(\frac{\mathrm{d}^2 u^{\mu}}{\mathrm{d}\tau^2} + \frac{u^{\mu}}{c^2} \frac{\mathrm{d}u_{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau} \right) \\ &= \frac{2}{3} K_{\mathrm{e}}\, q^2 \left(\frac{\mathrm{d}^3 x^{\mu}}{c^3 \mathrm{d}\tau^3} + \frac{\mathrm{d}x^{\mu}}{c\mathrm{d}\tau} \frac{\mathrm{d}^2 x_{\nu}}{c^2 \mathrm{d}\tau^2} \frac{\mathrm{d}^2 x^{\nu}}{c^2 \mathrm{d}\tau^2} \right) \\ &= \frac{2}{3} K_{\mathrm{e}}\, q^2 \left(\beth^{\mu} + \succapprox^{\mu} \backepsilon_{\nu} \backepsilon^{\nu} \right)\end{aligned}$$

Force (of radiation reaction):

$$\mathbf{F}_{\mathrm{rad}} \cdot \mathbf{v} = -\frac{2}{3} K_{\mathrm{e}} \frac{q^2}{c^3}\, a^2$$

"In an antenna, it is responsible for the radiation resistance."

Wikipedia:

Abraham-Lorentz force:

... [this] force ... requires the future to have an effect on the present, thus violating our intuition on the nature of cause and effect. ... In quantum electrodynamics, signals from the future are interpreted as anti-matter. The renormalization process fails, however, when applied to the gravitational force. ... Therefore general relativity has unsolved self-field problems. String theory is a current attempt to resolve these problems for all forces.

The Abraham-Lorentz force is therefore an entry point to some of the most perplexing mysteries of modern physics.

Check out Radiation resistance, and Radiation damping too...

11.2 Liénard-Wiechert (Retarded) Potentials [Gaussian Units]

A particle of charge q at position $r^{\mu}(\tau)$ and with 4-velocity $V^{\mu}(\tau)$ has the 4-potential

$$A^{\mu} = \frac{4\pi}{c} \int \mathrm{d}^4 x'\, D_{\mathrm{r}}(x - x')\, J^{\mu}(x')$$

where $D_{\mathrm{r}}(x - x')$ is the retarded Green function and

$$J^{\mu}(x') = qc \int \mathrm{d}\tau\, V^{\mu}(\tau)\, \delta^{(4)}[x' - r(\tau)]$$

This is the same as

$$\begin{aligned}\Phi(\mathbf{x},t) &= \left[\frac{q}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})R}\right]_{\text{ret}}\\ \mathbf{A}(\mathbf{x},t) &= \left[\frac{q\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})R}\right]_{\text{ret}}\end{aligned}$$

11.3 Fields of a Point Charge (in Arbitrary Motion) [Gaussian Units]

In general, when I use the phrase "fields of a point charge", it will be in this general context of arbitrary motion where the fields include static and radiation components.

(The following eqn is. . . ?)

$$F^{\alpha\beta} = \frac{q}{V\cdot(x-r)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{(x-r)^{\mu}V^{\beta}-(x-r)^{\beta}V^{\alpha}}{V\cdot(x-r)}\right]$$

where r^{α} and V^{α} are functions of τ. Now given that

$$\begin{aligned}(x-r)^{\alpha} &= (R,R\mathbf{n})^{\alpha} \qquad V^{\alpha}=(\gamma c,\gamma c\boldsymbol{\beta})\\ \frac{\mathrm{d}V^{\alpha}}{\mathrm{d}\tau} &= [c\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, c\gamma^2\dot{\boldsymbol{\beta}}+c\gamma^4\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})]\\ \tfrac{\mathrm{d}}{\mathrm{d}\tau}[V\cdot(x-r)] &= -c^2+(x-r)_{\alpha}\tfrac{\mathrm{d}V^{\alpha}}{\mathrm{d}\tau}\end{aligned}$$

this is the same as

$$\begin{aligned}\mathbf{E} &= q\left[\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\text{ret}}+\frac{q}{c}\left[\frac{\hat{\mathbf{n}}\times\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R}\right]_{\text{ret}}\\ \mathbf{B} &= [\hat{\mathbf{n}}\times\mathbf{E}]_{\text{ret}}\end{aligned}$$

11.4 Intensity and Power: Larmor and Liénard Formulas [Gaussian Units]

Intensity Distribution and Power of Radiation

In general,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\left|\mathbf{E}_{\text{accel}}\right|^2\hat{\mathbf{n}}$$

$$\frac{\partial^I 2}{\partial\omega\,\partial\Omega} = \frac{q^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\mathrm{d}t\;\hat{\mathbf{n}}\times[\hat{\mathbf{n}}\times\boldsymbol{\beta}]\,e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{r}(t)/c]}\right|^2$$

$$\frac{\partial^I 2}{\partial\omega\,\partial\Omega} = \frac{\omega^2}{4\pi^2c^3}\left|\int\mathrm{d}t\int\mathrm{d}x\;\hat{\mathbf{n}}\times[\hat{\mathbf{n}}\times\mathbf{J}(\mathbf{x},t)]\,e^{i\omega[t-\hat{\mathbf{n}}\cdot\mathbf{x}/c]}\right|^2$$

$$\begin{aligned}\frac{\mathrm{d}P(t')}{\mathrm{d}\Omega} &= R^2\,(\mathbf{S}\cdot\hat{\mathbf{n}})\tfrac{\mathrm{d}t}{\mathrm{d}t'} = R^2\,(\mathbf{S}\cdot\hat{\mathbf{n}})(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})\\ &= \frac{q^2}{4\pi c}\frac{\left|\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5}\end{aligned}$$

Relativistic Lorentz invariant generalization of the Larmor formula (Jackson pg 666)

$$\begin{aligned}P &= -\frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{\mathrm{d}p_{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau}\right)\\ &= \frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot{\boldsymbol{\beta}})^2-(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\right]\end{aligned}$$

since

$$-\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2 - \frac{1}{c^2}\left(\frac{\mathrm{d}E}{\mathrm{d}\tau}\right)^2 = \left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}\right)^2 - \beta^2\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}\right)^2$$

and $E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$. (From the fields of a charge in arbitrary motion given above, it is evident, Jackson claims, that the general result must involve only $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$, making this result unique. Might $\ddot{\boldsymbol{\beta}}$, jerk, be involved?)

Non-relativistic power distribution:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c}{4\pi}\left|R\mathbf{E}_{\text{accel}}\right|^2 = \frac{q^2}{4\pi c}\left|\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\boldsymbol{\beta}})\right|^2 = \frac{q^2}{4\pi c}\left|\dot{\mathbf{v}}\right|^2\sin^2\Theta$$

where Θ is the angle between the acceleration $\dot{\mathbf{v}}$ and $\hat{\mathbf{n}}$. Integrate to get the **Larmor Formula**:

$$P = \frac{2}{3}\frac{q^2}{c^3}\left|\dot{\mathbf{v}}\right|^2$$

Acceleration in the Direction of Motion

$$\frac{\mathrm{d}P(t')}{\mathrm{d}\Omega} = \frac{q^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

where θ is the angle between $\hat{\mathbf{n}}$ and $\boldsymbol{\beta}$ (or $\dot{\boldsymbol{\beta}}$).

12 Multipole Expansions (this includes energy expressions. . .)

12.1 General Ideas

- Inner/Outer (Spherical/Cylindrical) Multipole Expansions (wrt some coordinate system)
  - Expansion of the scalar potential in terms of multipole moments times spherical harmonics (wrt some origin & z-axis) divided by powers of the radial distance (wrt that origin)
  - Assumptions: charge density must either go to zero outside of some volume V or decrease faster than any polynomial as $r \to \infty$
    More assumptions?
  - Ideal vs. real multipoles (e.g. ideal vs. real dipole)
  - Explain how expansions change from one coord. system to another
  - Volume of interest (for Φ) vs. volume of integration V (for ρ)

$$\begin{aligned}
\frac{1}{R} &= \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\Theta}} \\
&\overset{1}{=} \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\Theta) \\
&\overset{2}{=} \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi') \\
&= \sum_{l,m}\frac{4\pi}{2l+1}\,\frac{{r_<}^l}{{r_>}^{l+1}}\,Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')
\end{aligned}$$

where Θ is the angle between $\mathbf{r} = \langle r, \theta, \phi \rangle_{\mathrm{s}}$ and $\mathbf{r}' = \left\langle r', \theta', \phi' \right\rangle_{\mathrm{s}}$ and $r_>$ and $r_<$ are the greater and lesser of r and r', respectively.
Equality 1: Legendre polynomial expansion of $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$
Equality 2: Addition Thm for spherical harmonics (wrt some origin & z-axis)

12.2 Electric expansion basics

Supposing the source charge $\rho(\mathbf{r})$ is localized (or dies off faster than any polynomial?) and the observation point is farther from the origin than (?),

$$\Phi(\mathbf{r}, t) = K_{\mathrm{e}} \sum_{lm} \frac{4\pi}{2l+1}\, q_{lm}(t)\, \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

since

$$\begin{aligned}
\Phi(\mathbf{x}, t) &= K_{\mathrm{e}} \int_V \rho(\mathbf{r}', t) \frac{1}{R}\, \mathrm{d}v' \\
&= K_{\mathrm{e}} \sum_{lm} \frac{4\pi}{2l+1} \left[\int_V \rho(\mathbf{r}', t)\, r'^l\, Y^*_{lm}(\theta', \phi')\, \mathrm{d}v' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \\
q_{lm}(t) &\equiv \int_V \rho(\mathbf{r}, t)\, r^l\, Y^*_{lm}(\theta, \phi)\, \mathrm{d}v
\end{aligned}$$

12.3 Magnetic expansion basics

Vector potential for a loop:

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= K_{\mathrm{m}} I \oint\limits_{\text{loop}} \frac{1}{R} \mathrm{d}\mathbf{l}' = K_{\mathrm{m}} I \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint\limits_{\text{loop}} (r')^n P_n(\cos\Theta') \mathrm{d}\mathbf{l}' \\
&= K_{\mathrm{m}} I \left[\frac{1}{r} \oint\limits_{\text{loop}} \mathrm{d}\mathbf{l}' + \frac{1}{r^2} \oint\limits_{\text{loop}} r' \cos\Theta' \mathrm{d}\mathbf{l}' + \frac{1}{r^3} \oint\limits_{\text{loop}} (r')^2 \left(\tfrac{3}{2}\cos^2\Theta' - \tfrac{1}{2} \right) \mathrm{d}\mathbf{l}' + \cdots \right] \\
&= K_{\mathrm{m}} I \left[0 + \frac{1}{r^2} \oint\limits_{\text{loop}} (\mathbf{r}' \cdot \hat{\mathbf{r}}) \mathrm{d}\mathbf{l}' + \cdots \right] \\
&= K_{\mathrm{m}} I \left[-\frac{1}{r^2}\, \hat{\mathbf{r}} \times \int_S \mathrm{d}\mathbf{a}' + \cdots \right] \\
&= K_{\mathrm{m}} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} + \cdots
\end{aligned}$$

where

$$\mathbf{m} \equiv I \int_S \mathrm{d}\mathbf{a} = I\mathbf{a}$$

12.4 Real Multipoles

Real Electric Dipole: Two Point-Charges

Real Magnetic Dipole: A Loop of Current

12.5 Multipole Expressions and Expansions

- Expressions independent of coordinate system

- Energy of localized distribution in an external field/potential $\Phi(\mathbf{r})$, $(\mathbf{A}(\mathbf{r}))$?; electric charge dist., magnetic multipole dist.:
  $W = \int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\,\mathrm{d}v$
  $W = \int_V ...\,\mathrm{d}v$

- Cartesian expressions:
  - Dipole moment (vectors); electric, magnetic:
    $\mathbf{p} = \int_V \mathbf{r}\,\rho(\mathbf{r})\,\mathrm{d}v$
    $\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}\times\mathbf{J}(\mathbf{r})\,\mathrm{d}v$
  - Traceless electric quadrupole moment tensor:
    $Q_{ij} = \int_V (3r_i r_j - r^2\delta_{ij})\,\rho(\mathbf{r})\,\mathrm{d}v$
    (Traceless? magnetic quadrupole moment tensor)?
  - Potential of a distribution; electric charge dist., magnetic multipole dist.:
    $\Phi(\mathbf{r}) = K_\mathrm{e}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{r_i r_j}{r^5} + \cdots\right]$
    (Potential of a magnetic multipole dist.)
  - Energy of a distribution; electric charge dist., magnetic multipole dist.:
    $W = q\,\Phi(\mathbf{0}) - \mathbf{p}\cdot\mathbf{E}(\mathbf{0}) - \frac{1}{6}\sum_{ij} Q_{ij}\,\frac{\partial E_j}{\partial x_i}(\mathbf{0}) + \cdots$
    $W = ... - \mathbf{m}\cdot\mathbf{B} - ...$
  - Field of a dipole at $\mathbf{x}_0$, where $\mathbf{R} = \mathbf{x} - \mathbf{x}_0$; electric, magnetic:
    $\mathbf{E}(\mathbf{x}) = -\boldsymbol{\nabla}(K_\mathrm{e}\frac{\mathbf{p}\cdot\hat{\mathbf{n}}}{R^2}) \;=\; K_\mathrm{e}\left[\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3} - \frac{4\pi}{3}\mathbf{p}\,\delta(\mathbf{R})\right]$
    $\mathbf{B}(\mathbf{r}) = -\boldsymbol{\nabla}(K_\mathrm{m}\frac{\mathbf{m}\cdot\hat{\mathbf{n}}}{R^2}) = K_\mathrm{m}\left[\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{R^3} + \frac{8\pi}{3}\mathbf{m}\,\delta(\mathbf{R})\right]$
    * Usually written as:
      $\mathbf{E}(\mathbf{x}) = K_\mathrm{e}\frac{3(\mathbf{p}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{p}}{R^3}$
      $\mathbf{B}(\mathbf{r}) = K_\mathrm{m}\frac{3(\mathbf{m}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}-\mathbf{m}}{R^3}$
    * Thm: Spherical volume enclosing charge (with dipole moment $\mathbf{p}$):
      $\int_{r<R}\mathbf{E}(\mathbf{x})\,d^3x = -\frac{\mathbf{p}}{3\varepsilon_0}$
    * Thm: Spherical volume w/o charge:
      $\int_{r<R}\mathbf{E}(\mathbf{x})\,d^3x = \frac{4\pi}{3}R^3\,\mathbf{E}(\mathbf{0})$
    * Magnetic Thms?
  - Energy of two (ideal) dipoles; electric, magnetic:
    $W_{12} = -\mathbf{p}_1\cdot\mathbf{E}_2 = K_\mathrm{e}\frac{\mathbf{p}_1\cdot\mathbf{p}_2-3(\mathbf{p}_1\cdot\hat{\mathbf{n}})(\mathbf{p}_2\cdot\hat{\mathbf{n}})}{R^3}$
    $W_{12} = -\mathbf{m}_1\cdot\mathbf{B}_2 = K_\mathrm{m}\frac{\mathbf{m}_1\cdot\mathbf{m}_2-3(\mathbf{m}_1\cdot\hat{\mathbf{n}})(\mathbf{m}_2\cdot\hat{\mathbf{n}})}{R^3}$
  - More?

Noteworthy

- The delta-function term in the magnetic dipole moment field expression is responsible for the hyperfine splitting in atomic spectra [see Griffiths pp 254, 157, D.J. Griffiths, *AM. J. Phys.* 50. 698 (1982)].

13 Basic Laws: Simple Geometries

13.1 Fields of a Point Charge in Linear Motion: Gaussian Units

(See Figure 11.9(b) in Jackson: Length contraction in the electric field. I should also include an image of the magnetic field.)

$$\begin{aligned}
\mathbf{E} &= \left(\frac{1}{\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}\right) q\,\frac{\hat{\mathbf{n}}}{R^2} \\
\mathbf{B} &= \boldsymbol{\beta}\times\mathbf{E}
\end{aligned}$$

$$\begin{aligned}
E_1 &= E_1' &&= -q\,\frac{\gamma(vt)}{\left(b^2+(\gamma vt)^2\right)^{3/2}} \\
E_2 &= \gamma E_2' &&= q\,\frac{\gamma b}{\left(b^2+(\gamma vt)^2\right)^{3/2}} \\
B_3 &= \gamma\beta E_2' &&= \beta E_2
\end{aligned}$$

(Ideally, I should do an analysis of a simple circular circuit and derive the magnetic field of the circuit in the limit as the particles become large in number and small in charge.)

Mention how this relates to length contraction and the Lorentz field transformation equations. Since

$$[(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})R]^2 = b^2+(vt)^2-(\beta b)^2 = \frac{1}{\gamma^2}(b^2+(\gamma vt)^2)$$

$$E_2 = \frac{e\gamma b}{\left(b^2+(\gamma vt)^2\right)^{3/2}} = e\left[\frac{b}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^3}\right]_{\text{ret}}$$

13.2 Fields of a Point Charge at Rest

$$\mathrm{d}\mathbf{E}(\mathbf{x}) = K_{\text{e}}\,\rho(\mathbf{x}')\,\frac{\hat{\mathbf{n}}}{R^2}\,\mathrm{d}^3x' \qquad \mathrm{d}\mathbf{E}(\mathbf{x}) = K_{\text{e}}\,\mathrm{d}q(\mathbf{x}')\,\frac{\hat{\mathbf{n}}}{R^2}$$

13.3 Electric Monopole Interaction (Coulomb Law, Electrostatics)

$$\mathbf{F}_{12} = K_{\text{e}}\frac{q_1q_2}{{r_{12}}^2}\hat{\mathbf{r}}_{12} \qquad \mathbf{F}_q = K_{\text{e}}\frac{qq'}{R^2}\hat{\mathbf{n}}$$

13.4 Magnetic Field Elements (Biot-Savart Law, Magnetostatics)

As a piece of an integral over a closed path (or a whole steady-state current distribution):

$$\mathrm{d}\mathbf{B}(\mathbf{x}) = K_{\text{m}}\,\mathbf{J}(\mathbf{x}')\times\frac{\hat{\mathbf{n}}}{R^2}\,\mathrm{d}^3x' \qquad \mathrm{d}\mathbf{B}(\mathbf{x}) = K_{\text{m}}\,\mathbf{K}(\mathbf{x}')\times\frac{\hat{\mathbf{n}}}{R^2}\,\mathrm{d}a' \qquad \mathrm{d}\mathbf{B}(\mathbf{x}) = K_{\text{m}}\,I\,\mathrm{d}\mathbf{s}(\mathbf{x}')\times\frac{\hat{\mathbf{n}}}{R^2}$$

((almost?) exact relativistic field, including acceleration effects)

For a point charge:

$$\mathbf{B}(\mathbf{x}) = K_{\text{m}}\,q\,\mathbf{v}(\mathbf{x}')\times\frac{\hat{\mathbf{n}}}{R^2}$$

(less exactly, doesn't include relativistic effects [Do Jackson 14.23 and 14.24 for subtle details], and, as Griffiths says on pg 219, this is "*approximately* right for nonrelativistice charges ($v \ll c$), under conditions where retardation can be neglected (see Ex. 10.4).")

Same, Jackson page 560:

$$\mathbf{B} \approx \frac{q}{c} \frac{\mathbf{v} \times \mathbf{r}}{r^3}$$

$$\Delta t \approx \frac{b}{\gamma v} (?)$$

How true is this equation (are these equations)?

13.5 Dipole Interactions

Electric

We can use $\mathbf{F}_{12} = \boldsymbol{\nabla}(\mathbf{p}_1 \cdot \mathbf{E}_2) = (\mathbf{p}_1 \cdot \boldsymbol{\nabla})\mathbf{E}_2 - \mathbf{m}_1 \times (\partial_t \cancel{\mathbf{B}_2})$ to derive the force $\mathbf{F}_{12}$ between two electric dipoles of moment $\mathbf{p}_1$ and $\mathbf{p}_2$, where their separation r_{12} is large compared to their sizes:

$$\begin{aligned}
F_{12} &\propto \frac{p_1 p_2}{{r_{12}}^4} \not\propto \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{{r_{12}}^2} \\
\mathbf{F}_{12} &= 3K_\mathrm{e} \frac{p_1 p_2}{{r_{12}}^4} \left[\left\{ \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2 - 5(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{12}) \right\} \hat{\mathbf{r}}_{12} + (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{p}}_1 + (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{p}}_2 \right] \\
\boldsymbol{\tau}_{12} &= K_\mathrm{e} \frac{p_1 p_2}{{r_{12}}^3} \left[3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{r}}_{12})(\hat{\mathbf{p}}_2 \times \hat{\mathbf{r}}_{12}) + (\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2) \right]
\end{aligned}$$

Note that $\boldsymbol{\tau}_{12} \neq -\boldsymbol{\tau}_{21}$ but that $\mathbf{r}_{12} \times \mathbf{F}_{12} + \boldsymbol{\tau}_{12} + \boldsymbol{\tau}_{21} = \mathbf{0}$, so total angular momentum is conserved by this dynamically isolated system.

…Torque and apparent non-conservation of momentum, etc.

Magnetic

We can use $\mathbf{F}_{12} = \boldsymbol{\nabla}(\mathbf{m}_1 \cdot \mathbf{B}_2) = (\mathbf{m}_1 \cdot \boldsymbol{\nabla})\mathbf{B}_2 + \mathbf{m}_1 \times (\mu_0 \cancel{\mathbf{J}_2} + c^{-2} \partial_t \cancel{\mathbf{E}_2})$ to derive the force $\mathbf{F}_{12}$ between two magnetic dipoles of moment $\mathbf{m}_1$ and $\mathbf{m}_2$, where their separation r_{12} is large compared to their sizes:

$$\begin{aligned}
F_{12} &\propto \frac{m_1 m_2}{{r_{12}}^4} \not\propto \frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{{r_{12}}^2} \\
\mathbf{F}_{12} &= 3K_\mathrm{m} \frac{m_1 m_2}{{r_{12}}^4} \left[\left\{ \hat{\mathbf{m}}_1 \cdot \hat{\mathbf{m}}_2 - 5(\hat{\mathbf{m}}_1 \cdot \hat{\mathbf{r}}_{12})(\hat{\mathbf{m}}_2 \cdot \hat{\mathbf{r}}_{12}) \right\} \hat{\mathbf{r}}_{12} + (\hat{\mathbf{m}}_2 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{m}}_1 + (\hat{\mathbf{m}}_1 \cdot \hat{\mathbf{r}}_{12})\hat{\mathbf{m}}_2 \right] \\
\boldsymbol{\tau}_{12} &= K_\mathrm{m} \frac{m_1 m_2}{{r_{12}}^3} \left[3(\hat{\mathbf{m}}_1 \cdot \hat{\mathbf{r}}_{12})(\hat{\mathbf{m}}_2 \times \hat{\mathbf{r}}_{12}) + (\hat{\mathbf{m}}_1 \times \hat{\mathbf{m}}_2) \right]
\end{aligned}$$

Note that $\boldsymbol{\tau}_{12} \neq -\boldsymbol{\tau}_{21}$ but that $\mathbf{r}_{12} \times \mathbf{F}_{12} + \boldsymbol{\tau}_{12} + \boldsymbol{\tau}_{21} = \mathbf{0}$, so total angular momentum is conserved by this dynamically isolated system.

…Torque and apparent non-conservation of momentum, etc. Hidden momentum in a magnetic dipole (see Griffiths pg 520).

Case	**Force $\mathbf{F}_{12}$**	**Torque $\boldsymbol{\tau}_{12}$**
$\hat{\mathbf{m}}_1 = \hat{\mathbf{m}}_2 = \hat{\mathbf{r}}_{12}$	$-6K_\mathrm{m} \frac{m_1 m_2}{{r_{12}}^4} \hat{\mathbf{r}}_{12}$ (attractive)	$\mathbf{0}$
$\hat{\mathbf{m}}_1 = \hat{\mathbf{m}}_2 \perp \hat{\mathbf{r}}_{12}$	$3K_\mathrm{m} \frac{m_1 m_2}{{r_{12}}^4} \hat{\mathbf{r}}_{12}$ (repulsive)	$\mathbf{0}$
$\hat{\mathbf{m}}_1 = -\hat{\mathbf{m}}_2 \perp \hat{\mathbf{r}}_{12}$	$-3K_\mathrm{m} \frac{m_1 m_2}{{r_{12}}^4} \hat{\mathbf{r}}_{12}$ (attractive)	$\mathbf{0}$

14 Media for Electromagnetic Phenomena

A vacuum (plural: vacua) is an absence of matter/material/media

14.1 General Media

- tenuous media, ponderable media
- Types: superconductor / perfect conductor, good conductor, poor conductor / semiconductor / poor insulator, good insulator, perfect insulator (specifications...)
  (A semiconductor may be an insulator when pure but a conductor when impure/doped, with the conductivity depending on the impurities and their quantity)
  Conductor ($\frac{\sigma(\omega)}{\omega} \gg \varepsilon(\omega)$), Dielectric ($\frac{\sigma(\omega)}{\omega} \ll \varepsilon(\omega)$), else General Media)
  dia: through; *dielectric*: substance or medium that transmits electric effects without conduction; non-conductor.
- (vacuum) (electric) permittivity $\underset{\sim}{\varepsilon}$, (magnetic) permeability $\underset{\sim}{\mu}$ (in general complex tensors) (impedance $\underset{\sim}{Z} = \sqrt{\mu/\underset{\sim}{\varepsilon}}$)
- conductivity σ (*not* charge area-density σ) (Complex $\underset{\sim}{\sigma}$ includes relaxation time effects), resistivity $\rho = \sigma^{-1}$ (*not* charge volume-density ρ)
- microscopic and macroscopic fields/eqns
- (in)homogeneity: homogeneous means having a common property throughout; a property is independent of position
- (an)isotropy: isotropic means having a common property in all directions, or along all axes; a property is invariant wrt direction (or rotation)

  Isotropic may imply a point in space (which all the said axes pass through), for instance, when one speaks of isotropic radiation (from a localized source)

  Isotropic can have the same meaning as homogeneous if, for instance, no central point for the axes is specified or implied (and the directions in question are all possible directions)
- (no) dispersion [$E = E(p)$ or $\omega = \omega(k)$]
- (non)linearity: linear media
- high-loss (lossy), low-loss, lossless (Jackson: . . . ε and μ real and positive – no losses)
  loss versus phase difference versus re-emission (reflection, refraction, etc)

14.2 Linear Media

Relate (refractive indices) with (dispersion relations) and (group velocities)

- "Complex Linear"?
- **P**: polarization, electric susceptibility χ_{e}, (relative) permittivity function $\varepsilon(\omega)$ (dielectric "constant") (complex)
- **M**: magnetization, magnetic susceptibility χ_{m}, (relative) permeability function $\mu(\omega)$ (complex)
  I have always seen it assumed that the material is "nonpermeable", i.e., $\mu = \mu_0 = (4\pi \times 10^{-7}\ \mathrm{N/A^2})$
- refractive index $\underset{\sim}{n}(\omega) = \sqrt{\mu_{\mathrm{r}}\, \underset{\sim}{\varepsilon}_{\mathrm{r}}(\omega)} = n_{\mathrm{n}} + i\kappa$ (n_{n}: "normal" index; κ: extinction coeff)
  . . .
- (angular) wave vector $\underset{\sim}{\mathbf{k}}$, $\underset{\sim}{\mathbf{k}}^2 = k^2\, \hat{\underset{\sim}{\mathbf{k}}} \cdot \hat{\underset{\sim}{\mathbf{k}}} = \mu\underset{\sim}{\varepsilon}\, \omega^2 = \underset{\sim}{n}^2\, (\omega/c)^2$ (a complex vector... ouch!)

- (angular) wavenumber $\underset{\sim}{k} = \sqrt{\mu\underset{\sim}{\varepsilon}}\,\omega = \underset{\sim}{n}\,(\omega/c) = \beta + i\alpha/2$ (angular spatial freq) (β: ?; α: attenuation constant or absorption coeff)
- A simple model for the dielectric function $\underset{\sim}{\varepsilon}(\omega)$ (appropriate for low density substances, small oscillations, and neglecting magnetic effects) which includes conductivity:

$$\underset{\sim}{\varepsilon}(\omega) = \varepsilon_0 + \frac{Ne^2}{m}\sum_j \frac{f_j}{{\omega_j}^2 - \omega^2 - i\omega\gamma_j}$$

Given that $\underset{\sim}{\mathbf{E}}(\mathbf{x}, t) = \mathbf{E}_\text{m}\, e^{i(\underset{\sim}{\mathbf{k}}\cdot\mathbf{x} - \omega t)}$ and each electron moves as $\underset{\sim}{\mathbf{x}}_j = \underset{\sim}{\mathbf{x}}_{j\text{m}}\, e^{i(\phi-\omega t)}$(?)
Quantum-mechanical definitions or phenomenological measurements of f_j (number of electrons per molecule with binding or resonant freq ω_j), ω_j, and γ_j (damping constants) make this an accurate description of the atomic contribution to the dielectric constant. N: molecule volume-density; Z: electron molecule-density; $\sum_j f_j = Z$. (Conduction electrons have a "resonant freq" of $\omega_0 = 0$.)
normal dispersion ($\text{Re}(\varepsilon)$ increases with ω) versus anomalous dispersion ($\text{Re}(\varepsilon)$ decreases with ω) and resonant absorption ($\text{Im}(\varepsilon)$ is large); ($\text{Im}(\varepsilon) < 0$ for lasers and masers)

  - Complex $\underset{\sim}{\varepsilon}(\omega) \Leftrightarrow$ Real $\varepsilon(\omega)$ (+ effect of damping forces) and $\sigma(\omega)$ (+ effect of relaxation time)
  - Low frequencies; electric conductivity – $\nabla\times\underset{\sim}{\mathbf{H}} = \underset{\sim}{\mathbf{J}}^\text{f} + \partial_t\underset{\sim}{\mathbf{D}} = (\underset{\sim}{\sigma} - i\omega\underset{\sim}{\varepsilon}_\text{b})\underset{\sim}{\mathbf{E}} = -i\omega(\underset{\sim}{\varepsilon}_\text{b} + i\frac{\underset{\sim}{\sigma}}{\omega})\underset{\sim}{\mathbf{E}}$
    $\underset{\sim}{\varepsilon}(\omega) \simeq \underset{\sim}{\varepsilon}_\text{b}(\omega) + i\frac{\underset{\sim}{\sigma}(\omega)}{\omega}$ ($\underset{\sim}{\varepsilon}_\text{b}$: the "normal" complex permittivity for the bound charges) From simple-model expression above we get Drude model conductance (essentially):

$$\underset{\sim}{\sigma}(\omega) = \frac{Ne^2}{m}\frac{f_0}{(\gamma_0 - i\omega)}$$

  - High frequencies; plasma freq – far above the highest resonant freq, $\varepsilon(\omega) \simeq \varepsilon_0 - \varepsilon_0\frac{{\omega_p}^2}{\omega^2}$, where

$${\omega_\text{p}}^2 = \frac{1}{\varepsilon_0}\frac{Ne^2}{m}Z = \frac{ne^2}{\varepsilon_0\, m}$$

    Then $ck = \sqrt{\omega^2 - {\omega_\text{p}}^2}$, yielding the dispersion relation $\underset{\sim}{\omega}^2 = {\omega_\text{p}}^2 + c^2k^2$
  - In dielectric media, only true for $\omega^2 \gg {\omega_\text{p}}^2$ ($\varepsilon_\text{r} \lessapprox 1$ and the wavenumber is real and varies with freq as for a mode in a waveguide with cutoff freq ω_p.)
  - In tenuous plasma (including/or ionosphere), true over wide range of frequencies, including $\omega < \omega_\text{p}$, where the wavenumber is purely imaginary and waves are reflected with fields inside falling off exponentially with distance from the surface. At $\omega = 0$, the attenuation constant $\alpha_\text{plasma} = 2\omega_\text{p}/c$
  - (I THOUGHT METALS WERE GOOD CONDUCTORS AND THEIR PERMITTIVITIES SHOULD BE MOSTLY IMAGINARY, I.E. HIGH (REAL) CONDUCTIVITY AND LOW REAL PERMITTIVITY) (I suppose then that that's true only for low frequencies, and at high frequencies, a conductor is like a (lossy/lossless?) dielectric. . .):
    In metals at optical and near, higher frequencies, reflectivity is caused by essentially the same behavior as for tenuous plasmas. For these frequencies, see the above "low freq" permittivity above. For $\omega \gg \gamma_0$, $\varepsilon(\omega) \simeq \varepsilon_b(\omega) - \varepsilon_0\frac{{\omega_\text{p}}^2}{\omega^2}$, where

$${\omega_\text{p}}^2 = \frac{ne^2}{\varepsilon_0\, m^*}$$

    is the plasma freq squared of the conduction electrons, given an effective mass m^* to include partially the effects of binding and n is electron volume-density ($NZ = n$ above). For $\omega \ll \omega_\text{p}$ waves are reflected and penetrate only a short distance into the metal (like for a plasma given. . .), but for large freq when $\varepsilon(\omega) > 0$, the metal can suddenly transmit.
    "ultraviolet transparency of metals"

- A better model for $\varepsilon(\omega)$ (using 4.69 $\chi_{\mathrm{e}} = N\gamma_{\mathrm{mol}}/(1 - \frac{1}{3}N\gamma_{\mathrm{mol}})$, where...):
- Poynting's Thm in Linear Dispersive Media w/ Losses
- Poynting's Thm for harmonic fields (Field definitions of Impedance $\underset{\sim}{Z}$ and Admittance $\underset{\sim}{Y} = \underset{\sim}{Z}^{-1}$)
- Kramers-Kronig Relations (or "dispersion relations")
  uses retarded auxiliary field $\mathbf{D}$, ... and a mathematical trick: complex freq $\underset{\sim}{\omega}$, and Cauchy's formula

$$\underset{\sim}{\varepsilon}(-\underset{\sim}{\omega}) = \underset{\sim}{\varepsilon}^*(\underset{\sim}{\omega}^*)$$

$$\underset{\sim}{\varepsilon}_{\mathrm{r}}(\omega) = 1 + \frac{1}{\pi i}\,\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}') - 1}{\underset{\sim}{\omega}' - \underset{\sim}{\omega}}\mathrm{d}\underset{\sim}{\omega}'\right]$$

  where $\mathcal{P}$ gives the principal part of the expression in brackets.

$$\mathrm{Re}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega})] = 1 + \frac{1}{\pi}\,\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Im}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}')]}{\underset{\sim}{\omega}' - \underset{\sim}{\omega}}\mathrm{d}\underset{\sim}{\omega}'\right]$$

$$\mathrm{Im}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega})] = -\frac{1}{\pi}\,\mathcal{P}\left[\int_{-\infty}^{\infty} \frac{\mathrm{Re}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}')] - 1}{\underset{\sim}{\omega}' - \underset{\sim}{\omega}}\mathrm{d}\underset{\sim}{\omega}'\right]$$

  OR

$$\mathrm{Re}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega})] = 1 + \frac{2}{\pi}\,\mathcal{P}\left[\int_{0}^{\infty} \frac{\omega'\,\mathrm{Im}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}')]}{\underset{\sim}{\omega}'^2 - \underset{\sim}{\omega}^2}\mathrm{d}\underset{\sim}{\omega}'\right]$$

$$\mathrm{Im}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega})] = -\frac{2\underset{\sim}{\omega}}{\pi}\,\mathcal{P}\left[\int_{0}^{\infty} \frac{\mathrm{Re}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}')] - 1}{\underset{\sim}{\omega}'^2 - \underset{\sim}{\omega}^2}\mathrm{d}\underset{\sim}{\omega}'\right]$$

  OR?...

$$\mathrm{Im}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\omega)] = -\frac{2\omega}{\pi}\,\mathcal{P}\left[\int_{0}^{\infty} \frac{\mathrm{Re}[\underset{\sim}{\varepsilon}_{\mathrm{r}}(\underset{\sim}{\omega}')] - 1}{\underset{\sim}{\omega}'^2 - \omega^2}\mathrm{d}\underset{\sim}{\omega}'\right]$$

Homogeneous Linear Media

- $\rho^{\mathrm{p}} \propto \rho^{\mathrm{n}}$

14.3 Conductors

- Perfect Conductor
  Just as in the static case, there is no electric field inside a perfect conductor. The charges inside a perfect conductor are assumed to be so mobile that they move instantly inresponse to changes in the fields, no matter how rapid, and always produce the correct surface-charge density $\Sigma = \hat{\mathbf{a}}\cdot\mathbf{D}$ to give zero electric field inside. Similarly, for time-varying magnetic fields, the surface charges move in response to the tangential magnetic field to produce always the correct surface current $\mathbf{K} = \hat{\mathbf{a}}\times\mathbf{H}$ to have zero magnetic field inside the perfect conductor. The other two boundary conditions are on normal $\mathbf{B}$ and tangential $\mathbf{E}$: $\hat{\mathbf{a}}_{21}\cdot(\mathbf{B}-\mathbf{B}_{\mathrm{c}}) = 0$, $\hat{\mathbf{a}}_{21}\times(\mathbf{E}-\mathbf{E}_{\mathrm{c}}) = \mathbf{0}$. From these boundary conditions we see that just outside the surface of a perfect conductor only *normal* $\mathbf{E}$ and *tangential* $\mathbf{H}$ fields can exist and that the fields drop abruptly to zero inside the perfect conductor.
- Real Conductor

- Quasi-Static Magnetic Fields (and their dispersion) in Conductors (See Jackson §5.18)
  "Quasi-static" refers to the regime for which the finite speed of light can be neglected and fields treated as if they propagated instantaneously. Said in other, equivalent words, it is the regime where the system is small compared with the electromagneic wavelength associiated with the dominant time scale of the problem. (... if the magnetic induction varies in time, an electric field is created, according to Faraday's law; the situation is no longer purely magnetic in character. Nevertheless, if the time variation is not too rapid, the magnetic fields dominate and the behavior can be called quasi-static.)
  * Skin depth, Eddy currents, Induction heating, Diffusion of magnetic fields in conducting media
  * **H** and **E** inside a conductor: rapid exponential decay, phase difference, and magnetic field "much larger than" the electric field. (See Jackson pp 354-355)
  * Skin depth
  $$\delta = \sqrt{\frac{2}{\mu_c \omega \sigma}}$$
    where the subscript c refers to the conductor
- Real BCs: use a successive approximation scheme
  (1) Assume that just outside the conductor there exists only a normal electric field $\mathbf{E}_\perp$ and a tangential magnetic field $\mathbf{H}_\|$, as for a perfect conductor. (The values of these fields are assumed to have been obtained from the solution of an appropriate boundary-value problem.)
  (2) Along with Maxwell's eqns, use the BC $\hat{\mathbf{a}}_{21} \times (\mathbf{H} - \mathbf{H}_c) = \mathbf{0}$ (since Ohm's law $\mathbf{J} = \sigma \mathbf{E}$ shows that with a finite conductivity there cannot actually be a surface layer of current as implied by $\hat{\mathbf{a}}_{21} \times \mathbf{H} = \mathbf{K}$) to solve for the fields within the transition layer and small coretions to the fields outside.
  (3) Use the fact that (or assume) the spatial variation of the fields normal to the surface is much more rapid than the variations parallel to the surface. (I.e., neglect all derivatives with respect to coordinates parallel to the surface compared to the normal derivative.)
  (4) Neglect the displacement current in the conductor.

$$\begin{aligned} \mathbf{H}_c &= \mathbf{H}_\| \, e^{-z/\delta} \, e^{iz/\delta} \\ \mathbf{E}_c &= \sqrt{\frac{\mu_c \omega}{2\sigma}} \, (1-i) \left(\hat{\mathbf{a}}_{21} \times \mathbf{H}_\| \right) \, e^{-z/\delta} \, e^{iz/\delta} \end{aligned}$$

where z is the coordinate perpendicular to the surface of the conductor, going inward

14.4 Nonlinear Media

- Pockels effect (birefringence...)
- Kerr effect (birefringence...)
- Hysteresis

15 Derivation of Macroscopic Equations via Atomic Theory

(Maybe trace the derivation, since derivations properly belong in "Derivation" papers.)

16 Boundary Conditions

See `Memorize_Equations.tex`

- Mathematical: Dirichlet, Neumann, Robin
- Media Interfaces
  - Ideal
    - Dielectric/Dielectric
    - Conductor/Dielectric
      Waveguide
      Resonant Cavities
      Optical Fibers
  - Realistic ... skin depth $\delta = \sqrt{\frac{2}{\mu\omega\sigma}}$

17 Reflection, Refraction, Surface Waves, and "Beam Tunneling"(?)

- Fresnel eqns, critical angle (total internal reflection), Brewster angle, Goos-Hänchen effect, etc.
  Reflection/Transmission Coefficients, Reflectance, Transmittance
- Surface Waves
    - waves guided by a gradient in refractive index
    - evanescent waves? (standing waves?): exponential decay, "fading away"
      An electromagnetic wave observed in total internal reflection, undersized waveguides, and in periodic dielectric heterostructures. Evanescent modes are characterized by an exponential attenuation and lack of a phase shift.
      Good animation: `http://users.ece.gatech.edu/~sungwon/f11.htm`
- Dissipation, exponential decay, etc.
- Frustrated Total Internal Reflection
- Eikonal approximation (?) ... quasi-geometrical optics ... $\boldsymbol{\nabla} S = n(\mathbf{x})\,\hat{\mathbf{k}}(\mathbf{x})$
- Generalization of Snell's law: $\mathrm{d}_s[n(\mathbf{r})\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}] = \boldsymbol{\nabla} n(\mathbf{r})$

18 Waveguides

- TEM waves; TM, TE, cutoff/evanescent modes; empty cylinder versus ("coaxial-type"?) waveguides
- Complex wavenumber: $k_\lambda \simeq k_\lambda^{(0)} + a_\lambda + i\beta_\lambda$
  where $k_\lambda^{(0)}$ is the wavenumber assuming perfectly conducting walls and β_λ is called the attenuation constant
  The energy flow along the guide: $P(z) = P_0 e^{-2\beta_\lambda z}$ (why?) so $\beta_\lambda = -\frac{1}{2P}\frac{\mathrm{d}P}{\mathrm{d}z}$

19 Special Categories...

19.1 Magnetostatics

$$\mathbf{E} = \mathbf{0}(?)$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\mathbf{A} = K_{\mathrm{m}} \int \mathrm{d}x' \, \frac{\mathbf{J}(\mathbf{x}')}{R} + \boldsymbol{\nabla}\Psi(\mathbf{x})$$

due to ambiguity: $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\Psi$ ("gauge freedom"?), where Ψ is a

19.2 Magnetism

- Magnets
- Ferromagnetism / Ferrimagnetism / Antiferromagnetism
- Spontaneous alignment (no external magnetic field)
- Paramagnetism
- Diamagnetism

19.3 Electretism?

- Electrets, Ferroelectric materials, hysteresis

20 Inductance, Capacitance, and Reactance; and Energy

Should I just list results and refer to the derivation file? (probably. or just list the important steps in the derivations)

20.1 Capacitance and Capacitors

$Q = CV$ Capacitance tells you the capacity of an (element) to hold charge, with respect to the voltage across that (element). (There does not have to be a current, circuit, or loop for a capacitor to hold charge – only an electric field.)

"Self Capacitance"?

- parallel plates
  (Explain why charge doesn't accumulate on the outer sides of the capacitor, why the fields basically stay inside the space between the plates, assuming they are close enough.)
- circular-cylindrical shell
- spherical shell
- ...

20.2 Inductance and Inductors

What kinds of inductive effects are possible? How does one determine the boundaries for inductive phenomena?

SEE JACKSON §5.17 and around there… learn approximation methods
(look at §5.5 too)

Mutual Inductance

- Continuous distribution:
- Two inductors: $N_1\Phi_{12} = M_{12}I_2$; $N_2\Phi_{21} = M_{21}I_1$
- Multiple inductors:

Self-Inductance

$\Phi = LI$, $V_{\mathrm{emf}} = L\frac{\mathrm{d}I}{\mathrm{d}t}$ Inductance tells you

- straight wire
- solid circular cylinder
  - current following axis of symmetry
  - current circulating about axis of symmetry
- circular-cylindrical shell
  - current circulating about axis of symmetry (idealized solenoid)
    (Explain why the field is essentially zero outside of the solenoid.)
  - current following axis of symmetry
- …
- spherical shell (say, non-conducting but holding charge and rotating?)
- …

20.3 Reactance and Reactors(?)

- Inductive reactance
- Capacitive reactance

20.4 Energy

$$W_{\mathrm{e}} = \tfrac{1}{2}\int_{\rho\text{ dist.}} \Phi\,\rho\,\mathrm{d}^3x = \tfrac{\varepsilon_0}{2}\int_{\text{all space}} E^2\,\mathrm{d}^3x$$

$$W_{\mathrm{m}} = \tfrac{1}{2}\int_{\mathbf{J}\text{ dist.}} \mathbf{A}\cdot\mathbf{J}\,\mathrm{d}^3x = \tfrac{1}{2\mu_0}\left[\int_V B^2\,\mathrm{d}^3x - \oint_S (\mathbf{A}\times\mathbf{B})\cdot\mathrm{d}\mathbf{a}\right] = \tfrac{1}{2\mu_0}\int_{\text{all space}} B^2\,\mathrm{d}^3x$$

Hey, wouldn't it be nicer if there was a plus sign in front of the surface integral term (so that it represents the energy in the magnetic field outside of the region V)? Switch **A** and **B** to make it so, but also perform a calculation to prove that it's a positive amount.

- Where's the energy? (Poynting's Thm, etc)

21 Radiation in General

21.1 Types and Sources of Radiation/Scattering

Types

- Non-Electromagnetic Radiation
  - [Composition]: particles other than photons that radiate (i.e., are propelled away from a source). E.g., nuclear radiation; hadron, lepton, or guage particle radiation; graviton or gravitational radiation)
  - [Composition]: nonphotonic nonparticulate objects or energy that radiate(s)
- Electromagnetic Radiation: (the energy of this radiation is called radiant energy)
  - [Composition]: "photons"; elctric and magnetic fields
  - [Source]: A charged object moving due to a dynamic, radiating em-field (radiation)
    - Scattering
      - Thomson Scattering (localized source) (Raleigh, Mie, Wigglers?, etc?)
      - Reflection/Refraction/Transmission Diffraction Diffusion (Diffuse Refl/Refr/Trans)
  - [Source]: A charged object moving due to a dynamic, nonradiating em-field
    - (Radio broadcast)
  - [Source]: A charged object moving in a(n approximately) fixed em-field (could be ∼no field)
    - Cyclotron Radiation: due to (non-relativistic) motion in a magnetic field
      - Charged particle whizzes around in a circular orbit (or helix) and radiates
      - (!) The planet Jupiter in particular is a large source of cyclotron radiation.
    - Synchrotron Radiation: due to ultrarelativistic motion in a magnetic field
      - a.k.a. Ivanenko and Pomeranchuk radiation(?)
    - Cherenkov Radiation: "shockwave radiation"
      - A photonic shockwave front caused by a superluminal charged particle
      - Results when a charged particle, most commonly an electron, exceeds the (phase) speed of light in a dielectric (electrically insulating) medium through which it passes.
      - Like a sonic boom or the envelope of the wake of a boat (that's travelling faster than surface wave speeds)
      - Smith-Purcell Effect, anomolous Cherenkov effects
    - Bremsstrahlung radiation: "braking radiation" (or, more generally, "acceleration radiation")
      - (from German *bremsen*, to brake, and *Strahlung*, radiation)
      - Strictly speaking, bremsstrahlung refers to any radiation due to the acceleration of a charged particle, which includes synchrotron radiation; however, it is frequently used in the more literal and narrow sense of radiation from electrons stopping in matter.
      - a.k.a. "free-free radiation" (why?)
  - [Property]: (in)coherent
  - [Property]: (circularly, elliptically, non/un)polarized

  - [Property]: (mono,poly)chromatic
  - [Effect]: (non)ionizing

Sources

(also "types"?)

- Incandescence (ideally, Black Body Radiation)
- Photoluminescence
- Fluorescence (a.k.a. Luminescence or Photoluminescence)
- Phosphorescence
- Electroluminescence
- Chemoluminescence
- Triboluminescence
- Bioluminescence
  - Shrimpoluminescence
- Sonoluminescence

  Scintillation (a flash of light produced in a transparent material by an ionization event)
  Radiation from excitation-decay spectra
  LASER

21.2 Radiating Systems, Multipole Fields

- Near (Static) Zone
- Intermediate (Induction) Zone
- Far (Radiation) Zone

21.3 Dipoles in Arbitrary Motion(?)

22 Scattering, Diffraction, and Collisions

Say how this relates to radiation...

22.1 Scattering and Diffraction

Thomson Scattering

Thomson formula, Thomson cross section, classical electron radius

The Thomson differential cross section for charged particles (area/solid angle) is

$$\sigma \equiv \left(\frac{q^2}{mc^2}\right)^2 = \left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right)^2$$

Compton Scattering

Klein-Nishina formula (Jackson pg 697)

22.2 Other Kinds of Scattering

From QED

- Delbrück scattering (has been observed)
  The deflection of high-energy photons in the Coulomb field of nuclei as a consequence of vacuum polarization.
- light-light scattering (has not been observed)

22.3 Viewing Reflection, Refraction, and in terms of Scattering and Diffraction

22.4 Collisions, Energy Loss, and Scattering of Charged Particles, Cherenkov and Transition Radiation

23 Models, Paradoxes, Catastrophes, and Controversies

- Photon Model
- Other Particles
  - Fluxions (electric fluxions and magnetic fluxions) are the constituent particles of electric and magnetic flux. (Not to be confused with Newton's fluxions, or derivatives, and fluents, or continuous functions.) The electric fluxions have a spin of $\frac{1}{2}$ and the magnetic fluxions have a spin of $-\frac{1}{2}$.
- Abraham-Minkowski controversy

24 Phenomena and Applications

- Methods of separating charge
  - Contact Electrification
    electronegativity, electron affinity
    Triboelectric effect, Friction machines
  - Electrostatic induction (Electrodynamic induction?)
    Influence machines
  - Electrochemistry
    Batteries (cells, piles)
  - Piezoelectric effect
  - Thermoelectric effect
    Pyroelectric effect, Peltier effect, Seebeck effect, Thomson effect
- Sky's Colors
  - Rayleigh scattering (elastic scattering by particles much smaller than the wavelength of the light), Mie scattering (by spherical particles), Tyndall effect (scattering by particles in colloid systems, such as suspensions or emulsions)
    * Contrasted with Rayleigh scattering, Mie solutions to scattering include all possible ratios of diameter to wavelength.
    * Rayleigh scattering increases linearly with the fourth power of frequency
  - Raman scattering (inelastic scattering where the frequency of the light shifts)
- Sun's Color: Why white or yellow?

- Optical Depth: absorption, tranmission, (absorptivity, etc. versus absorbance, transmittance in spectroscopy); Beer-Lambert law
- Critical Opalescence
  - Critical opalescence is a phenomenon in liquids close to their critical point, in which a normally transparent liquid appears milky due to density fluctuations at all possible wavelengths.
  - Due to large density fluctuations. Related (quantitatively) to Rayleigh scattering.
- Plasma Frequency: what does it tell us? what's interesting about it?
  It tells us when a EM wave will be reflected or transmitted through a material.
  (Know how to derive this, and, as Nathan Kugland said, derive Faraday rotation too while I'm at it.)
- Water's Blue: See `http://en.wikipedia.org/wiki/Permittivity`
- Telegraph lines (Heaviside's telegraph eqns, etc.)
- Antennae
- Radiation Sources with "insertion devices" (e.g. wigglers and undulators) (see Jackson pg 661 and §14.7)
- Aerodynamics and Magnetostatics
  - The Biot-Savart law is also used to calculate the velocity induced by vortex lines in aerodynamic theory. (The theory is closely parallel to that of magnetostatics; vorticity corresponds to current, and induced velocity to magnetic field strength.)
- Evanscent waves
  - Dark field, scattering microscopy
  - Scanning near-field optical microscopy
  - Beamsplitter
  - Fiber-optic coupling
  - Finger-print scanner

24.1 Electrical Tools and Measuring Devices

- Electroscope
- Oscilloscope
- Ammeter
- Voltmeter
- Multimeter

25 Magnetic Monopoles

26 Open Questions and Mysteries

27 Eponymous and Esoteric Terms

Eponyms

- Gauss's and Faraday's Laws and the Ampère-Maxwell Law
- Ohm's "Law"
- Kirchoff's Laws (Kirchoff integral)
- Hertz vector (polarization potential) (see Jackson pg 280)
- Joule Heating Law
- Liénard-Wiechert Potential
- Kramers-Kronig Relations
- Brewster angle
- Darwin Lagrangian (Breit interaction)
- Proca Lagrangian (Proca Eqns; photon mass effects?)
- Superconductivity (London penetration depth)
- Poynting's Thm (Poynting vector)
- Green's Thm (Green functions)
- BMT equation (V. Bargmann, L. Michel, and V. L. Telegdi)
  - A more elegant look at the equation of motion for spin (using Lorentz covariance)
- Thomas precession (see Jackson Section 11.8)
- Thomson scattering of radiation (see Jackson pg 694)
- Klein-Nishina formula (for Compton scattering) (See Jackson)
- FitzGerald-Lorentz contraction
- Lorentz factor γ
- Claussius-Mossotti equation →(optical frequencies)→ Lorenz-Lorentz equation
- Heaviside-Feynman expressions (for fields of a point charge)
- Jefimenko's generalizations (of the Coulomb and Biot-Savart Laws)
- Cherenkov radiation (Čerenkov)
- (quantum) Hall effect, Hall voltage

Esoterica

- Pinch effect

28 Mathematical Reference

28.1 Special Function Facts

28.2 Green Function Facts?

28.2.1 Convention 1: Mathematics Green Functions

$$\nabla'^2 G = \delta$$

$$\Phi = -4\pi \int_V G\,\rho\,\mathrm{d}v' - \oint_S \left[G\tfrac{\partial\Phi}{\partial n'} - \tfrac{\partial G}{\partial n'}\Phi\right]\mathrm{d}a'$$

In detail:

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

$$\Phi(\mathbf{r}) = -4\pi \int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}v' - \oint_S \left[G(\mathbf{r},\mathbf{r}')\,\tfrac{\partial\Phi}{\partial n'}(\mathbf{r}') - \tfrac{\partial G}{\partial n'}(\mathbf{r},\mathbf{r}')\,\Phi(\mathbf{r}')\right]\mathrm{d}a'$$

28.2.2 Convention 2: Electrostatics Green Functions

$$\nabla'^2 G = -4\pi\,\delta$$

$$\Phi = K_\mathrm{e} \int_V G\,\rho\,\mathrm{d}v' + \frac{1}{4\pi}\oint_S \left[G\tfrac{\partial\Phi}{\partial n'} - \Phi\tfrac{\partial G}{\partial n'}\right]\mathrm{d}a'$$

In detail:

$$\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\,\delta(\mathbf{r}-\mathbf{r}')$$

$$\Phi(\mathbf{r}) = K_\mathrm{e} \int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}v' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{r},\mathbf{r}')\,\tfrac{\partial\Phi}{\partial n'}(\mathbf{r}') - \tfrac{\partial G}{\partial n'}(\mathbf{r},\mathbf{r}')\,\Phi(\mathbf{r}')\right]\mathrm{d}a'$$

For Dirichlet BCs, where Φ is known on the boundary S, let

$$G_\mathrm{D}(\mathbf{r},\mathbf{r}') = 0 \qquad \text{for } \mathbf{r}' \text{ on } S$$

so

$$\Phi(\mathbf{r}) = K_\mathrm{e} \int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}v' - \frac{1}{4\pi}\oint_S \frac{\partial G_\mathrm{D}}{\partial n'}(\mathbf{r},\mathbf{r}')\,\Phi(\mathbf{r}')\,\mathrm{d}a'$$

For Neumann BCs, where $\partial\Phi/\partial n$ is known on the boundary S, let

$$\frac{\partial G_\mathrm{N}}{\partial n'}(\mathbf{r},\mathbf{r}') = -\frac{4\pi}{A_S} \qquad \text{for } \mathbf{r}' \text{ on } S$$

so

$$\Phi(\mathbf{r}) = \langle\Phi\rangle_S + K_\mathrm{e} \int_V G(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}v' + \frac{1}{4\pi}\oint_S G(\mathbf{r},\mathbf{r}')\,\frac{\partial\Phi}{\partial n'}(\mathbf{r}')\,\mathrm{d}a'$$

References

[1] John David Jackson: *Classical Electrodynamics, Third Edition*, John Wiley and Sons, Inc. (1999)

[2] David J. Griffiths: *Introduction to Electrodynamics, Third Edition*, Prentice Hall (1999)

[3] Michael E. Peskin, Daniel V. Schroeder: *An Introduction to Quantum Field Theory*, Westview Press (1995)