Notes on Trigonometry
Andrew Forrester January 28, 2009

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1 The Big Picture

Trigonometry (or “trig”) is a branch of mathematics that deals with the relationships between aspects of triangles and circles (the angles and lengths of sides of triangles and a special set of triangles that relate to the unit circle) and the abstraction of these relationships to elementary periodic functions, triangles on spherical surfaces and spherical (solid) angles, (and complex functions?).

Planar trigonometry, spherical trigonometry, (complex trigonometry?).
(Further generalizations are in what branches of math?)

Some early uses of the word trigonometry, found in the Oxford English Dictionary:


2 Definitions

Given\(1\) Dom(arcsin) = \([-1, 1]\), Cod(arcsin) = \([-\frac{\pi}{2}, \frac{\pi}{2}]\); Dom(arccos) = \([-1, 1]\), Cod(arccos) = \([0, \pi]\); Dom(arctan) = \((−\infty, \infty)\), Cod(arctan) = \([-\frac{\pi}{2}, \frac{\pi}{2}]\),

and given that we want Dom(arcsin2) = \(\mathbb{R}^2\), Cod(arcsin2) = \([0, 2\pi]\); Dom(arccos2) = \(\mathbb{R}^2\), Cod(arccos2) = \([0, 2\pi]\); and Dom(arctan2) = \(\mathbb{R}^2\), Cod(arctan2) = \([0, 2\pi]\),

\[
\begin{align*}
\text{arcsin2}(y, x) &:= \begin{cases} 
\arcsin(y/x), & \text{for } x \geq 0 \text{ and } y \geq 0 \\
\pi - \arcsin(y/x), & \text{for } x < 0 \\
2\pi + \arcsin(y/x), & \text{for } x \geq 0 \text{ and } y < 0
\end{cases} \\
\text{arccos2}(y, x) &:= \begin{cases} 
\arccos(y/x), & \text{for } x \geq 0 \\
2\pi - \arccos(y/x), & \text{for } x < 0
\end{cases} \\
\text{arctan2}(y, x) &:= \begin{cases} 
\arctan(y/x), & \text{for } x \geq 0 \text{ and } y \geq 0 \\
\pi - \arctan(y/x), & \text{for } x < 0 \\
2\pi + \arctan(y/x), & \text{for } x \geq 0 \text{ and } y < 0
\end{cases}
\end{align*}
\]

\(1\)Dom\((f)\) is the domain of the function \(f\) and Cod\((f)\) is the codomain of \(f\), also sometimes called the range or target.
3 Fundamental Rules

• Pythagorean Identity
  \[ a^2 + b^2 = c^2 \]

• Trigonometric Pythagorean Identity
  \[ \sin^2 \theta + \cos^2 \theta = 1 \]

• Law of Cosines
  \[ a^2 + b^2 = c^2 + 2ab \cos \gamma \]

• Law of Sines
  \[ \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R \]
  where \( R \) is the radius of the circumcircle (the smallest circle containing the triangle)

• Law of Tangents
  \[ \frac{a + b}{a - b} = \tan \left[ \frac{1}{2} (\alpha + \beta) \right] \]

4 Addition and Multiplication Rules

(How to derive these?)

• Angle Addition/Subtraction Identities
  \[
  \begin{align*}
  \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \sin \beta \cos \alpha \\
  \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \beta \sin \alpha
  \end{align*}
  \]

  – Double-Angle Identities
  – Half-Angle Identities
  – “Phase constant removal”
  \[
  \begin{align*}
  A \sin(kx \pm \phi) &= A \sin kx \cos \phi \pm A \sin \phi \cos kx \
  B \cos(kx \pm \phi) &= B \cos kx \cos \phi \mp B \sin \phi \sin kx
  \end{align*}
  \]
  
  – “Phase . . . ”
  \[
  \begin{align*}
  R \sin kx \pm S \cos kx &= A \sin(kx \pm \phi) \\
  &= \sqrt{R^2 + S^2} \sin (kx \pm \arctan2(S, R)) \\
  \mp (-S \cos kx \mp R \sin kx) &= \mp B \cos(kx \pm \varphi) \\
  &= \pm \sqrt{R^2 + S^2} \cos (kx \pm \arctan2(-R, S))
  \end{align*}
  \]

* If \( R = A \cos \phi \) and \( S = A \sin \phi \), then \( S/R = \tan \phi \) so
  \[
  \begin{align*}
  \phi &= \arctan2(S, R) \\
  \sin \phi &= \sin \left( \tan^{-1}(S/R) \right) = \frac{S}{\sqrt{R^2 + S^2}} \\
  \cos \phi &= \cos \left( \tan^{-1}(S/R) \right) = \frac{R}{\sqrt{R^2 + S^2}} \\
  A &= R/ \cos \phi = S/ \sin \phi = \sqrt{R^2 + S^2}
  \end{align*}
  \]
* If \(-S = B \cos \varphi\) and \(R = B \sin \varphi\), then \(-R/S = \tan \varphi\) so
  \[
  \varphi = \arctan2(-R, S)
  \]
  \[
  \sin \varphi = -R/\sqrt{R^2 + S^2}
  \]
  \[
  \cos \varphi = S/\sqrt{R^2 + S^2}
  \]
  \[
  B = -S/\cos \varphi = R/\sin \varphi = -\sqrt{R^2 + S^2}
  \]

- **Sum-to-Product Formulas**

  \[
  \sin \alpha + \sin \beta = \]

- **Product-to-Sum Formulas**

  \[
  \sin \alpha \sin \beta = \]

5 Integration