Notes on Probability and Statistics
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1 The Big Picture

• Statistics is a mathematical science pertaining to the collection, analysis, interpretation or explanation, and presentation of data (i.e., information or facts).
  – It may be descriptive, inferential, or mathematical (concerned with the theoretical basis of the subject).
  – The term originated as the German word Statistik, denoting analysis of data about the German state and signifying the “science of state”.
  – A statistic is an individual datum, so statistics may also refer to the data in question.

• Probability theory deals with the potential truth of statements or potential actuality of events and the underlying mechanics of the relevant systems that determine truth or reality.
  – Probability is the likelihood that something is the case or will happen.
  – Probability theory is used in statistics, mathematics, science, philosophy, etc.
  – Two broad interpretations of probability:
    * Frequency Probability: deals only with well-defined random experiments. The relative frequency of occurrence of an experiment’s outcome, when repeating the experiment, is a measure of the probability of that random event.
    * Bayesian Probability: probabilities can be assigned to any statement whatsoever, even when no random process is involved, as a way to represent its subjective plausibility.

• Randomness
2 Counting with Combinatorics

2.1 Possibly Useful Notation

(Q=Question, A=Answer)

• n-set

• k-sub-n-set (and k-combination, or k-subset)
  Q: How many k-sub-n-sets are there?
  A: \( n \text{ choose } k \).

• k-multi-n-set (and “k-multiset”)
  Q: How many k-multi-n-sets are there?
  A: \( n \text{ multichoose } k \).

• k-permu-n-set (and “k-permutation”)
  (“permuchoose”?)

• Lists? (n-list, k-sub-n-list, etc?)

partitioning...

Variations \( \leftrightarrow \) k-permutations
<table>
<thead>
<tr>
<th>Permutation ⇒ (ordered) multilist</th>
<th>Combination ⇒ (non-ordered) multiset</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Permutation</strong> of an n-set</td>
<td></td>
</tr>
<tr>
<td>(of a set ⇒ no reps)</td>
<td>n-permutation</td>
</tr>
<tr>
<td>Q: How many n-permutations are there?</td>
<td>A: <em>n</em> permuchoose <em>n</em>.</td>
</tr>
<tr>
<td><em>P</em>(<em>n</em>, <em>n</em>) = ( \frac{n!}{(n-n)!} = n! )</td>
<td></td>
</tr>
<tr>
<td><strong>Permutation w/o reps</strong> from an n-set</td>
<td>k-perm-n-set</td>
</tr>
<tr>
<td>Q: How many k-perm-n-sets are there?</td>
<td>A: <em>n</em> permuchoose <em>k</em>.</td>
</tr>
<tr>
<td><em>P</em>(<em>n</em>, <em>k</em>) = ( \frac{n!}{(n-k)!} )</td>
<td></td>
</tr>
<tr>
<td><strong>Combination w/o reps</strong> from an n-set</td>
<td>k-sub-n-set</td>
</tr>
<tr>
<td>Q: How many k-sub-n-sets are there?</td>
<td>A: <em>n</em> choose <em>k</em>.</td>
</tr>
<tr>
<td><em>C</em>(<em>n</em>, <em>k</em>) ≡ ( \binom{n}{k} ) = ( \frac{n!}{k!(n-k)!} )</td>
<td></td>
</tr>
<tr>
<td><strong>Permutation with reps</strong> from an n-set</td>
<td>k-multiperm-n-set</td>
</tr>
<tr>
<td>Q: How many k-multiperm-n-sets are there?</td>
<td>A: <em>n</em> multipermuchoose <em>k</em>.</td>
</tr>
<tr>
<td><em>MP</em>(<em>n</em>, <em>k</em>) = <em>n</em>^k</td>
<td></td>
</tr>
<tr>
<td><strong>Permutation with reps</strong> from an (n, p, . . . , q)-multiset</td>
<td>k-multiperm-(n, p, . . . , q)-multiset</td>
</tr>
<tr>
<td>Q: How many k-multiperm-(n, p, . . . , q)-multisets are there?</td>
<td>A: (<em>n</em>, p, . . . , q) multipermuchoose (<em>k</em>, l, . . . , m).</td>
</tr>
<tr>
<td><em>MP</em>[(n, p, . . . , q), (k, l, . . . , m)] = ( C(N, k)C(N - k, l) \cdots C(N - k - l - \cdots , m) )  ( = \frac{N!}{k! \cdots m!} )</td>
<td>where ( N = k + l + \cdots + m ) and ( k \leq n, l \leq p, \ldots, m \leq q ).</td>
</tr>
<tr>
<td><strong>Combination with reps</strong> from an n-set</td>
<td>k-multi-sub-n-set</td>
</tr>
<tr>
<td>Q: How many k-multi-sub-n-sets are there?</td>
<td>A: <em>n</em> multichoose <em>k</em>.</td>
</tr>
<tr>
<td><em>MC</em>(<em>n</em>, <em>k</em>) = ( \frac{(n + k - 1)!}{k!(n-1)!} = \binom{n + k - 1}{k} )</td>
<td>(Check with Griffiths’ Intro to Quantum, pg 235).</td>
</tr>
<tr>
<td><strong>Combination with reps</strong> from an (n, p, . . . , q)-multiset</td>
<td>k-sub-(n, p, . . . , q)-multiset</td>
</tr>
<tr>
<td>Q: How many k-sub-(n, p, . . . , q)-multisets are there?</td>
<td>A: (<em>n</em>, p, . . . , q) multichoose <em>k</em>.</td>
</tr>
<tr>
<td><em>MC</em>[(n, p, . . . , q), <em>k</em>] =</td>
<td></td>
</tr>
<tr>
<td><strong>Partition</strong></td>
<td></td>
</tr>
<tr>
<td>Q:</td>
<td>A:</td>
</tr>
<tr>
<td>( \binom{n \text{ choose } k} = \frac{(n \text{ permuchoose } k)}{[(# \text{ of possible orderings of a } k\text{-list}) = (k \text{ permuchoose } k)]} )</td>
<td></td>
</tr>
</tbody>
</table>
\[ MP((n,p,\ldots,q),(k,l,\ldots,m)) = C(N,k)C(N-k,l)\cdots C(N-k-l-\ldots,m) \]
\[ = \frac{N!}{k!(N-k)!l!(N-k-l)!\cdots m!(N-k-l-\ldots-m)!} \]
\[ = \frac{N!}{k!(N-k)!l!(N-k-l)!\cdots m!} \]

2.2 Distinguish
- combination and permutation
- multi-subset and sub-multiset
- multicomb and multiperm

2.3 Permutation (without Repetition)
In this combinatorial context, a permutation is a sequence of distinct objects. A permutation from a set is a sequence without repetitions that contains elements from that set. (Is there an “r-permutation”?) (In algebra, permutation may be a bijection of a set into itself.) The number of \( r \)-permutations from a set of \( n \) elements is
\[ P(n,r) \equiv \frac{n!}{(n-r)!} \]
Rephrasing, this is the number of sequences you can make with \( r \) non-repeating elements from a set of \( n \) elements.
\[ P(n,r) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-r)!} \]
Note that
\[ P(n,n) = \frac{n!}{0!} = n! \]

2.4 Combination (Without Repetition), Binomial Coefficients
In combinatorial mathematics, a combination is simply a subset. (As sets are not ordered and each element of a set is unique, having multiplicity of 1, combinations are also un-ordered and contain only unique elements.) A \( k \)-combination (or \( k \)-subset) is a subset with \( k \) elements. Given a set of \( n \) elements, the number of possible \( k \)-combinations (all with a fixed number \( k \) of elements), called “\( n \) choose \( k \)”, is
\[ C(n,k) \equiv \binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \]
This is also a binomial coefficient. Binomial Theorem:
\[ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k \]
This can be thought of loosely as the number of permutations, minus ordering. Each \( k \)-permutation is a \( k \)-combination that is arranged in a list. Some permutations have the same elements, but arranged in different orders. For each \( k \)-combination, there are \( P(k,k) = k! \) \( k \)-permutations. To find the number of
In $k$-combinations, we just have to divide the number of $k$-permutations by $k!$, the number of $k$-permutations per $k$-combination:

$$C(n, k) = \frac{\text{(Number of } k\text{-permutations)}}{\text{(Number of } k\text{-permutations per } k\text{-combination})} = \frac{P(n, k)}{P(k, k)} = \frac{n!}{k!(n-k)!}$$

Try this:  
$$C(n, k) = \frac{\text{(Number of } k\text{-subset-lists of an } n\text{-set)}}{\text{(Number of } k\text{-lists})}$$

<table>
<thead>
<tr>
<th>Combination Example</th>
<th>Combination Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>You have a set of five objects ${A, B, C, D, E}$ and you'd like to know how many ways you can choose distinguishable 3-combinations from this set.</td>
<td>$C(5, 3) = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4}{2} = 5 \cdot 2 = 10$</td>
</tr>
<tr>
<td>You have 10 empty (distinguishable, or numbered) positions and you’d like to know how many distinguishable ways you can place 3 indistinguishable objects (3 red marbles, for example) in these positions. (This is similar to having 10 distinguishable objects and wondering how many ways you can pick 3-combinations from this set.)</td>
<td>$C(10, 3) = \frac{10!}{3!(10-3)!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 5 \cdot 3 \cdot 8 = 120$</td>
</tr>
<tr>
<td>How many $k$-sub-$n$-sets are there?</td>
<td>$n \text{ choose } k$</td>
</tr>
</tbody>
</table>

Pascal’s Formula(?) - This may come in handy:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

### 2.5 Permutation with Repetition, from a Set

$$n^r$$

### 2.6 Permutation with Repetition, from a Multiset, Multinomial Coefficients

$$\binom{N}{k_1, k_2, \ldots, k_s} = \frac{N!}{k_1! k_2! \cdots k_s!}$$

Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_s)^N = \sum_{k_1, k_2, \ldots, k_s} \binom{N}{k_1, k_2, \ldots, k_s} x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}$$

### 2.7 Combination With Repetition

Multisets... the number of $k$-multisets?... Given a set $S = \{A, B, C\}$ and a multiplicity function $m : S \to \mathbb{Z}^+$, you can construct and represent a multiset by $M \equiv (S; m)$, an ordered pair of a set and a multiplicity function on that set into the positive integers.

Maybe you could also have a multiplicity function $m : S \to \mathbb{N}$, but then you have to be more careful when comparing multisets.
Anyway a \( k \)-multiset would then be a multiset \( M \equiv (S;m) \) such that \( \sum_{s \in S} m(s) = k \), so there are \( k \) elements in the set, of various possible multiplicities.

The multisets \( \{A, B, C\} \) and \( \{B, A, C\} \) are equal, but \( \{A, A, B, C\} \neq \{A, B, C\} \).

“\( n \) multichoose \( k \)”

\[
\frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}
\]

**Stars and Bars**

You have a set of \( n = 5 \) objects \( \{A, B, C, D, E\} \) and you’d like to know how many ways you can choose 6-multisets (\( k = 6 \)) from this set. Here’s a few possible distinguishable 6-multisets from our “\( n \)-set”:

\[
\begin{align*}
\ast\ast & | \ast | \ast | \ast & = & \{A, A, B, C, D, E\} \\
\ast | \ast & \ast & \ast & \ast & = & \{A, B, C, C, D, E\} \\
\ast & | \ast & \ast & | \ast & = & \{A, B, C, D, E, E\} \\
\| & \ast & \ast & \ast & \ast & = & \{C, C, C, E, E, E\} \\
| & \ast & \ast & \ast & \ast & \ast & | & = & \{B, B, B, B, B, B\}
\end{align*}
\]

So we see we have 6 “stars” representing the \( k \) elements of the \( k \)-multisets and \( n-1 \) “bars” representing the partitions between distinguishable elements. We also see that the positions of the stars and bars are distinguishable, but the bars (or stars) are indistinguishable from each other. So this is just like choosing positions for \( n-1 \) indistinguishable objects among \((n-1)+k\) distinguishable positions, which we showed (well, we haven’t shown it yet...) is similar to having \((n-1)+k\) distinguishable objects and wondering how many ways you can pick \((n-1)\)-combinations from this set. That’s just “\((n-1)+k\) choose \((n-1)\)”.

<table>
<thead>
<tr>
<th>Combination With Repetition Example</th>
<th>Combination With Repetition Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>You have a set of five objects ( {A, B, C, D, E} ) and you’d like to know how many ways you can choose 6-multisets from this set.</td>
<td>( C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5 \cdot 4}{2} = 5 \cdot 2 = 10 )</td>
</tr>
<tr>
<td>How many ( k )-multi-( n )-sets are there?</td>
<td>( n ) multichoose ( k )</td>
</tr>
</tbody>
</table>

2.8 **Partitioning**

The number \( N \) of ways of partitioning \( n \) distinct objects into \( k \) distinct groups containing \( n_1, n_2, ..., n_k \) objects, respectively, where each object appears in exactly one group and \( \sum_{i=1}^{k} n_i = n \), is

\[
N = \binom{n}{n_1, n_2, \ldots, n_k} \equiv \frac{n!}{n_1! n_2! \cdots n_k!}
\]

Multinomial coefficients.

3 **Geometry and Probability**

See Table 1.
Table 1: Analogies between geometry and probability (from [1] pg 424)

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length squared of vector</td>
<td>Variance</td>
</tr>
<tr>
<td>Length of vector</td>
<td>Standard deviation</td>
</tr>
<tr>
<td>Dot product</td>
<td>Covariance</td>
</tr>
<tr>
<td>Cosine of angle between two vectors</td>
<td>Correlation</td>
</tr>
</tbody>
</table>

4 Statistics

- Random Variable
  \[ X \]

- Expected Value
  \[ E(X) \]

- Moment

- Covariance
  \[ \text{cov}(X,Y) = E((X - \mu)(Y - \nu)) \]
given \( E(X) = \mu \) and \( E(Y) = \nu \)

- Correlation:
The correlation coefficient \( \rho_{X,Y} \) between two random variables \( X \) and \( Y \) with expected values \( \mu_X \) and \( \mu_Y \) and standard deviations \( \sigma_X \) and \( \sigma_Y \) is defined as:
  \[ \rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E((X - \mu_X)(Y - \mu_Y))}{\sigma_X \sigma_Y} \]

- Coherence: like correlation, coherence gives a measure of the dependence of two random variables.
  \[ \gamma^2 = \frac{E(|XY|^2)}{E(|X|^2)E(|Y|^2)} \]

5 Theorems

- Central Limit Thm

6 Sampling

Measured Values “True” Values Theoretical Values Calculated Values

- Set Multiset Sequence
- Population Sample Variable Random Variable (continuous, discrete) Ensemble
  (Arithmetic) Mean Expected Value (of a Random Variable) Median (minimum, first quartile, median, third quartile, maximum) Mode

- Sample Mean Biased sample or Unbiased sample Population Mean (“true mean”) Sample Variance (or Standard Deviation) Biased estimator or Unbiased estimator Population Variance (or Standard Deviation)

  Deviation Error (usually unobservable) (Deviation from the population mean) Residual (observable) (Deviation from the sample mean) Average Deviation (Mean Absolute Deviation) Moments about the
Origin (nth moment) Moments about the Mean (nth central moment) Standardized Moments (nth standardized moment) = \[ \text{[nth central moment]} / \text{[standard deviation]}^n \] Variance (2nd), Skewness (3rd), Kurtosis (4th) (Pearson skewness coefficients different from skewness) (Kurtosis Excess different from kurtosis) Generalized Mean (Power Mean, M(t) generalized mean with exponent t) (minimum, harmonic mean, geometric mean, arithmetic mean, quadratic mean, maximum) (Square-Root) Root Mean Square (RMS) (Quadratic Mean) Standard Deviation (RMS Deviation from the Mean) Standard Deviation of the Mean Generalized f-Mean

Entropy

Histogram (discrete data) Probability Density Function Cumulative Distribution Function Characteristic Function Moment-Generating Function Cumulant-Generating Function (Logarithm of the moment-generating function) Cumulant Problem of Moments (Statistical) Bias (Unbiased Estimator)

Uncertainty Error

Accuracy Veracity (What is “true” anyway?) Precision Repeatability (of measurements over a short period of time using the same equipment and operator). Reproducibility (of measurements over a long period of time using different equipment and operators). False Precision Bias (not statistical bias?) (...in calibration)

Significant Figures Scientific Notation Other Notation Significance Arithmetic Propagation of Error Partial Differentials Propagation of Error Formula Change of Units Change of Base

Thought: If you measure the length of something and see that it is between 4 and 5 mm long and reaches 2/3 across, you could say that it is 4.6 mm or 4.7 mm across. But mustn’t you also know how precise your ruler is? And how precise was the machine that made the ruler? I think this relates to the idea of traceability to a standard of reference. The standard is defined to be accurate (but it may change over time...).

References
