Notes on Group Theory
Lie Groups, Lie Algebras, and Linear Representations
Andrew Forrester     January 28, 2009

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1 Physics 231B

Subjects: Lie groups, Lie algebras, their representation theory and perhaps also some differential geometry of Lie groups and homogeneous spaces.

2 Questions

2.1 Basic Questions

• Where does the exponential map come from, and why can any element be written as

\[ g = \exp \left( \sum_i t_i x_i \right) \]

where the \( t_i \) are the parameters associated with the generators \( x_i \)?

– Is this true?: I think that the exponential map comes up because we have a multiplication \((g \cdot h)\) and exponentiation \((g^n)\), but since exponentiation is continuous, we want an easy or uniform way to talk about fractional and real exponents (rather than merely integer exponents):

\[
\begin{align*}
  a &= a^1 = e^x \\
  a^b &= e^c \\
  g &= g^1 = \exp(x) \\
  g^r &= \exp(y)
\end{align*}
\]

\[
\begin{align*}
  x &= \ln a, \quad a, x \in \mathbb{R} \\
  c &= b \ln a \\
  x &= \left. \frac{d}{dx} g^\lambda \right|_{\lambda=0} \\
  y &= r x
\end{align*}
\]
• Are the generators a basis for the Lie Algebra? (In HW3, if \{H, H^+, H^-\} are a basis, then how can I write I as a linear combination of these elements?)

• What is a representation? How do “roots and weights” come into play?

• What does a spinor have to do with spin?
  (One thing has to rotate or cycle; the spinor transforms due to this rotation, it comes to its negative after one full rotation, and it comes back to its original state after two full rotations.)
  (Perhaps this actually depends on the parametrization of the “rotating” thing, since it may be that the angles go like \(\pi \rightarrow 2\pi, \pm\pi/2\ldots\) see group theory homework #2.)

2.2 Other Questions

• Why can \(SU(2)\) be Cartan-decomposed? (What does Cartan decomposition mean?)

2.3 Things to look into

• Lie groups provide a natural framework to analyze continuous symmetries of differential equations (Picard-Vessiot theory), much in the same way as permutation groups are used in Galois theory to analyze discrete symmetries of algebraic equations.

3 Elementary Group Theory

3.1 Vocabulary

• Binary Operation
• Associativity
• Closed
• Identity
• Algebraic Spaces
  • Semi-Group
  • Group
  • Subgroup
  • Supergroup

3.2 Theorems

4 Lie Groups and Lie Algebras

4.1 Groups

• Topological (Continuous) Group
  A topological group \(G\) is a that is also a group where the

• Parametric Group

• Lie (Smooth) Group
  A Lie group \(G\) is a smooth manifold that is also a group where the
4.2 Algebras

- **Algebra**
  
  An algebra is a vector space with a(n associative, bilinear) multiplication (which can be written as a bracket).

- **Lie Algebra**
  
  A Lie algebra is an algebra where the bracket (multiplication) satisfies
  
  - antisymmetry
  - the Jacobi identity

  More generally, the antisymmetry property of the bracket is replaced by the (nilpotent) property, and the (nilpotent) property along with bilinearity give antisymmetry.

5 Linear Representations of (Lie) Groups and (Lie) Algebras

5.1 Stuff

- **Diagrams**
  
  - Weight diagram
  - Root diagram?
  - Young Tableaux
  - Dynkin diagram (for ... , for simple roots)

  Defining representation, fundamental representation, adjoint representation, General representations, “there are \( n \) fundamental reps”

  Rank of a Lie algebra, Generators, Cartan subalgebra, (center), Diagonalizable generators (weights), Cartan generators, off-diagonal generators (roots), weight vector (different conventions?), root vector, root = weight for adjoint rep (roots are intrinsic to the algebra, weights?), highest weight

  positive, negative, simple roots

  constructing a representation = constructing all weights, structure relations?

  No canonical ordering of the roots, partial ordering

  \[ \lambda = \sum_i q_i \lambda_i \] (?)

5.2 From Abers

\[
D(g_1) D(g_2) = D(g_1 g_2) \\
\bar{g} = e^{\bar{X}} \\
D(\bar{g}) = D(e^{\bar{X}}) = e^{\bar{D}(\bar{X})} \\
\bar{D}(\bar{X}_1) \bar{D}(\bar{X}_2) \neq \bar{D}(\bar{X}_1 \bar{X}_2) \\

[J^i, J^j] = i\hbar \varepsilon^{ijk} J^k \quad \mathbf{J} \times \mathbf{J} = i\hbar \mathbf{J} \\
[L^i, L^j] = i\hbar \varepsilon^{ijk} L^k \quad \mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}
\]
\[ [S^i, S^j] = i\hbar \varepsilon^{ijk} S^k \quad S \times S = i\hbar S \]

\[ \left[ \frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i\varepsilon^{ijk} \sigma^k \]

\[ S = \hbar \frac{\sigma}{2} \]

\[ D^{(j)}(\bar{R}) = \exp \left( -i \theta \sum_i n_i D^{(j)}(\bar{J}_i) \right) \]

The collection of matrix elements of the \( D^{(j)}(\bar{R}) \) with the conventional phases are the Wigner matrices.

\[ D^{(1/2)}(\bar{J}_i) = \frac{\hbar}{2} \sigma_i \quad \text{or} \quad \frac{1}{2} \sigma_i \text{ in Abers' notation (not due to natural units).} \]

\[
D^{(1)}(\bar{J}_z) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

\[
D^{(1)}(\bar{J}_+) = \begin{pmatrix}
0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2} \\
0 & 0 & 0
\end{pmatrix}
\]

\[
D^{(1)}(\bar{J}_-) = D^{(1)}(\bar{J}_+) \dagger
\]

\[
R(\hat{n}, \theta) = e^{-i\hat{n} \cdot J} = e^{-i\hat{n} \cdot L} e^{-i\hat{n} \cdot S}
\]

where

\[
e^{-i\hat{n} \cdot L} |r, m\rangle = |r', m\rangle
\]

\[
e^{-i\hat{n} \cdot S} |r, m\rangle = \sum_{m'} |r, m'\rangle D(\hat{n}, \theta)_{m'm}
\]

The \( J^i \)'s must obey the same commutation rules as the \( L^i \)'s for the commutation rules reflect the law of combination of rotations and must be obeyed by any triplet of generators (the consistency condition) whatever be the nature of wave function they rotate. So

\[ L \times L = i\hbar L \quad \Rightarrow \quad J \times J = i\hbar J \]
\[ J^\pm = J^1 \pm iJ^2 \]
\[ [J^i, J^j] = i\varepsilon^{ijk} J^k \]
\[ [J, J^+] = \]
\[ [J, J^-] = \]
\[ [J^+, J^-] = 2\hbar J^3 \]
\[ [J^3, J^\pm] = \pm\hbar J^\pm \]
\[ [J^2, J^\pm] = 0 \]
\[ [L^3, L^\pm] = \pm\hbar L^\pm \]
\[ [L^2, L^\pm] = 0 \]
\[ [L^2, L^i] = 0 \]
\[ [H, L^i] = 0 \]
\[ [H, L^2] = 0 \]

\[ J^i J^j = ? \]

\[ L^\pm \rightarrow \pm \hbar e^{\pm i\phi} (\partial_\theta \pm i \cot \theta \partial_\phi) \]

\[ L^2 = r^2 p^2 - (r \cdot p)^2 + i\hbar r \cdot p \]

Spherical Harmonics as Rotation Matrices: see Sakurai pg 202