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0 Section Numbering

• If numbering = 0, there is no numbering.
• If numbering ≠ 0, there is numbering.

numbering = 1

1 Index Gymnastics

Let \( \delta_{ab} = \delta^a_b \), \( \delta_{abc} = \delta^a_b \delta^c_c \), and so forth.

Theorem: 
\[ \sum_i \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \]
\[ \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{lm} - \delta_{ml} \]

Proof: Since
\[ \varepsilon_{ijk} = \varepsilon_{ija} \delta^a_l \]
\[ = \varepsilon_{iab} (\delta_{aj} - \delta_{bj}) \]
\[ = \varepsilon_{abc} (\delta^a_{ij} - \delta^a_{jk}) + \delta^a_{ij} - \delta^a_{jk} + \delta^a_{jk} - \delta^a_{ij} \]

1
\[ \varepsilon^{ijk} \varepsilon_{ilm} = \varepsilon^{iab} (\delta_{jk}^{ab} - \delta_{ba}^{jk}) \varepsilon_{icd} (\delta_{lm}^{cd} - \delta_{mc}^{dl}) \]
\[ = \text{in-between steps?} \]
\[ = \delta_{jk}^{ik} - \delta_{ml}^{ik} \]

Theorem:
\[ \sum_{ij} \varepsilon_{ijk} \varepsilon_{ijl} = 2 \delta_{kl} \]
\[ \varepsilon_{ijk} \varepsilon_{ijl} = 2 \delta_{l}^{k} \]

Proof:

2 Special Relativity Transformations

2.1 Notation
This all has to be justified, proved, and explained geometrically.
\[ x^\mu' = x^{\prime \mu} \]
No, that won’t work because it implies
\[ x'^\mu = x^\mu \]
How about
\[ (x^\mu)' = x'^\mu ? \]

2.2 Time Dilation
Imagine a light-clock with one photon bouncing back and forth... (maybe start by assigning the time \( \Delta \tau \) to each tick)

- \( l \) = the path length for a photon in one cycle of the stationary clock \((l = 2h)\)
- \( l' \) = the path length for a photon in one cycle of the moving clock
- \( \Delta t \) = time for one cycle (“tick”) of the moving clock, as measured by the stationary clock
- \( \Delta \tau \) = the time for one cycle (“tick”) of the moving clock, as measured by the moving clock
  - \( \Delta t \) and \( \Delta \tau \) are measured in the same units
  - (Looking at \( \Delta t = N \Delta \tau \), one can see that \( \Delta \tau \) is the unit given to each tick of the stationary clock.) \((2h \text{ is one light-} \Delta \tau: 2h = c \Delta \tau.\))
- \( N \) = the number of cycles/ticks of the stationary clock during one cycle/tick of the moving clock
- \( N' \) = the number of cycles/ticks of the moving clock during one cycle/tick of the moving clock \((N' = 1)\)

![Figure 1: Stationary Lab Light-Clock and Moving (Length-Contracted) Light-Clock](image)
In the time it takes for a photon to traverse the path of length \(l\) for one cycle of the stationary clock, a photon in the moving clock will only have gone part of the way for one cycle in the moving clock (a fraction of \(l'\)), so a moving clock will run slow relative to a stationary clock. (Note that in the moving clock’s frame, the “stationary” clock will run slow compared to the “moving” clock.)

\[
\frac{\Delta t}{\Delta \tau} = \frac{N}{N'} = N = \frac{l'}{l} = \frac{2\sqrt{\left(\frac{1}{2}v\Delta t\right)^2 + h^2}}{2h} \quad (1)
\]

\[
\Rightarrow \quad 2h\Delta t = 2\sqrt{\left(\frac{1}{2}v\Delta t\right)^2 + h^2} \Delta \tau \quad (2)
\]

\[
\Rightarrow \quad 4h^2(\Delta t)^2 = 4\left[\frac{1}{4}v^2(\Delta t)^2 + h^2\right] (\Delta \tau)^2 \quad (3)
\]

\[
\Rightarrow \quad \left(\frac{2h}{c}\right)^2(\Delta t)^2 = \left[v^2(\Delta t)^2 + \left(\frac{2h}{c}\right)^2\right] (\Delta \tau)^2 \quad (4)
\]

\[
\Rightarrow \quad (\Delta t)^2 = \left[v^2(\Delta t)^2 + (\Delta \tau)^2\right] \quad (5)
\]

\[
\Rightarrow \quad (1 - v^2/c^2)(\Delta t)^2 = (\Delta \tau)^2 \quad (6)
\]

\[
\Rightarrow \quad \pm\sqrt{1 - v^2/c^2} \Delta t = \Delta \tau \quad (7)
\]

\[
\Rightarrow \quad \sqrt{1 - v^2/c^2} \Delta t = \Delta \tau \quad (8)
\]

\[
\Rightarrow \quad \Delta t = \gamma_v \Delta \tau \quad (9)
\]

• Now, if we redefine \(\Delta t\) to be some duration measured by a stationary clock, then \(\Delta \tau = \Delta t/\gamma_v\) will be the corresponding duration measured by clock moving at a speed \(v\) relative to the stationary clock

• Next, one should answer the question: “Why should the stationary light-clock time be the ‘actual’ time in a reference frame?” Does it have to do with the fact that EM interactions determine the time-scale of most events? Could other interactions determine other time scales? Or are the other interactions also based on (massless) particles that move at the speed \(c\)?

• \(\tau\) is the proper time (“self-time”, “rest-time”, “intrinsic time”) of the moving light-clock (in other words, the time according to a moving clock)

• \(t\) is the “lab time” measured by a stationary light-clock

### 2.3 Length Contraction

• \(L_0\) is the length of the light-clock in the “lab frame” when it is stationary, or the length of the moving light-clock in its rest frame

• \(L\) is the length of the moving light-clock in the “lab frame”

• \(l\) = the path length for a photon in one cycle of the stationary clock \((l = 2L_0)\)

• \(l'\) = the path length for a photon in one cycle of the moving clock

• \(\Delta t\) = time for one cycle (“tick”) of the moving clock, as measured by the stationary clock
  - \(\Delta t\) is the time for the photon to go back and forth once in the moving clock
  - \(\Delta t_1\) is the time for the photon to go in the direction \(\hat{v}\) (in the same direction as the clock’s motion) and cover a distance \(l_1'\)
- $\Delta t_2$ is the time for the photon to go in the direction $-\hat{v}$ (in the direction opposite the clock’s motion) and cover a distance $l'_2$.
- $\Delta t = \Delta t_1 + \Delta t_2$ and $l' = l'_1 + l'_2$.

• $\Delta \tau$ is the time for one cycle (“tick”) of the moving clock, as measured by the moving clock.
  - $\Delta t$ and $\Delta \tau$ are measured in the same units.
  - (Looking at $\Delta t = N \Delta \tau$, one can see that $\Delta \tau$ is the unit given to each tick of the stationary clock.) ($2L_0$ is one light-$\Delta \tau$: $2L_0 = c\Delta \tau$.)

• $N = \text{the number of cycles/ticks of the stationary clock during one cycle/tick of the moving clock}$

• $N' = \text{the number of cycles/ticks of the moving clock during one cycle/tick of the moving clock}$ ($N' = 1$)

Figure 2: Stationary Lab Light-Clock and Moving Length-Contracted Light-Clock

\[
\begin{align*}
l'_1 & = L + v\Delta t_1 = c\Delta t_1 \\
l'_2 & = L - v\Delta t_2 = c\Delta t_2 \\
\Delta t & = \Delta t_1 + \Delta t_2 \\
& = \frac{L}{c-v} + \frac{L}{c+v} \\
& = \frac{L(c+v)}{c^2-v^2} + \frac{L(c-v)}{c^2+v^2} \\
& = \frac{2L}{c} \frac{1}{1-v^2/c^2} \\
& = \gamma_v^2 \frac{2L}{c} \\
\frac{2L_0}{c} & = \Delta \tau \\
& = \frac{1}{\gamma_v} \Delta t \\
& = \frac{1}{\gamma_v} \left( \gamma_v^2 \frac{2L}{c} \right) \\
& = \gamma_v \frac{2L}{c} \\
\Rightarrow \quad L_0 & = \gamma_v L
\end{align*}
\]
2.4 Lorentz Transformation

Make this more complete, general, rigorous:

Two observers, one in an inertial frame $K$ and the other in another inertial frame $K'$, measure the position of the wavefront of a pulse of light emitted from a point source. They agree that their frame origins coincide when the pulse of light is emitted (Not that the event of the emission of the pulse of light in spacetime is their common origin... think about that...) $R^2 = c^2 t^2 = x^2 + y^2 + z^2$ and $R'^2 = c^2 t'^2 = x'^2 + y'^2 + z'^2$...

\[-c^2 t^2 + x^2 + y^2 + z^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2 = 0\]

This event (point source emission of light) could happen at any point in spacetime and the observers will agree

\[-c^2 t^2 + x^2 + y^2 + z^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2 = s^2\]

Call this invariant $s$ of inertial frame/coordinate change the spacetime interval and we have (after more work)

\[s^2 = \eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu\]

\[ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu\]

\[\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]

Later

\[ds^2(x^\alpha) = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu\]

So if we want to find the possible transformations $\Lambda^\mu'_{\ \mu}$ from $x^\mu$ to $x'^\mu$

\[x'^\mu = \Lambda^\mu'_{\ \mu} x^\mu\]

\[s^2 = \eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu'\nu'} x'^\mu x'^\nu = \eta_{\mu'\nu'} \Lambda^\mu'_{\ \mu} \Lambda^\nu'_{\ \nu} x^\mu x^\nu\]

\[\eta_{\mu'\nu'} \Lambda^\mu'_{\ \mu} \Lambda^\nu'_{\ \nu} = \eta_{\mu\nu}\]

\[
\ldots
\]

General form

and

\[\Lambda = \begin{pmatrix} \gamma & -\gamma \beta x \\ -\gamma \beta x & \gamma \end{pmatrix}\]

\[\Lambda = \begin{pmatrix} \gamma & \gamma \beta \\ \gamma \beta & \gamma \end{pmatrix}\]

(passive versus active?)
2.5 Inverse Lorentz Transformation

2.6 Four-Derivatives

- Four-Derivatives
  \[ \partial_{\mu}' = \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\mu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\mu}} \]

- Four-Gradient
- Directional Derivative
- Four-Divergence
  \[ \partial_{\mu} \phi(x^{\alpha}) = \frac{\partial}{\partial x'^{\mu}} \phi(x^{\alpha}) \]

- d’Alembertian (Four-Laplacian)
  \[ \partial_{\mu} \phi(x^{\alpha}) \partial^{\mu} \phi(x^{\alpha}) = \frac{\partial}{\partial x'^{\mu}} \phi(x^{\alpha}) \cdots \]

- Four-Curl

2.7 Addition of Velocity (Three-Velocity Change of Coordinates)

In a reference frame \( K \), the velocity of an object is measured to be \( \vec{u} \). In another reference frame \( K' \), which is moving in the \( x \) direction with a velocity \( v_x \) with respect to \( K \), that object’s velocity is given by

\[
\begin{align*}
  u'_x & = \frac{dx'}{dt'} = \frac{dx^1'}{1} \frac{dt^0'}{c} = \frac{\gamma_v (dx^1 - \beta_x dx^0)}{c \gamma_v dt - \beta_x dx} = \frac{dx^1 - \beta_x c dt}{c \gamma_v (dt - \beta_x dx)} = \frac{u_x - v_x}{1 - \frac{v_x u_x}{c^2}} \\
  \vec{u}'_{yz} & = \frac{d\vec{x}'_{yz}}{dt'} = \frac{d\vec{x}_{yz}}{c dt'} = \frac{\gamma_v (dx^1 - \beta_x dx^0)}{c \gamma_v dt - \beta_x dx} = \frac{\gamma_v (dt - \beta_x dx)}{c \gamma_v (dt)} = \frac{\gamma_v (1 - \frac{v_x u_x}{c^2})}{c \gamma_v (1 - \frac{\vec{u} \cdot \vec{u}}{c^2})}
\end{align*}
\]

In other notation

\[
\begin{align*}
  (u')^\parallel & = \frac{u^\parallel - v}{1 - \frac{u \cdot v}{c^2}} \\
  (u')^\perp & = \frac{u^\perp}{\gamma_v (1 - \frac{u \cdot v}{c^2})}
\end{align*}
\]

2.8 Addition of Momenta (Relativistic Three-Momentum Change of Coordinates)

...
3 Momentum

Maybe I should do fundamental transformations first (time dilation, length contraction, Lorentz transformation), then momentum (and maybe other stuff), and then other transformations.

- Start with Momentum is conserved, and Lorentz transformations
- End with Derive $p = \gamma v m$

Use http://en.wikibooks.org/wiki/Special_Relativity:_Dynamics and look at other papers (In “Look” folder)

4 Energy

(I define a parameter $\eta$, a.k.a. the relativistic $\beta$, since it seems to be a useful notation and mnemonic for these energy expressions and for the Lorentz tranformation equations. If $\eta$ is defined as $\frac{p}{mc}$, then it would be ideal for the Hamiltonian view, while $\beta = \frac{v}{c}$ is ideal for the Lagrangian view. The only drawback seems to be that the letter $\eta$ is already being used for the Minkowski metric, so one would have to be careful about that, or use some other letter.)

$$\beta = \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\eta \equiv \gamma \beta = \gamma \frac{v}{c} = \frac{u}{c} = \frac{p}{mc}$$

$$\gamma = (1 - \beta^2)^{-1/2} = (1 + \eta^2)^{1/2}$$

since

$$\gamma^2 = \frac{1}{1 - \beta^2} = \frac{1 - \beta^2 + \beta^2}{1 - \beta^2} = 1 + \gamma^2 \beta^2$$

$$0 \leq \beta < 1$$

$$1 \leq \gamma < \infty$$

$$0 \leq \eta < \infty$$

$$\beta = \left(1 - \frac{1}{\gamma^2}\right)^{1/2}$$

$$\eta = \left(\gamma^2 - 1\right)^{1/2}$$

$$\gamma = \frac{E}{mc^2}$$
\[ E = \gamma mc^2 \]
\[ = (1 - \beta^2)^{-1/2} mc^2 \]
\[ = (1 + \eta^2)^{1/2} mc^2 \]
\[ = \left( m^2 c^4 + \eta^2 m^2 c^4 \right)^{1/2} = \left( m^2 c^4 + \left( \frac{p}{mc} \right)^2 m^2 c^4 \right)^{1/2} = \left( m^2 c^4 + p^2 c^2 \right)^{1/2} \]
\[ E^2 = m^2 c^4 + p^2 c^2 \]

\[ E = \sqrt{m^2 c^4 + p^2 c^2} \]
\[ = mc^2 \sqrt{1 + \eta^2} \]
\[ = mc^2 \left[ 1 + \frac{1}{2} \eta^2 - \frac{1}{8} \eta^4 + \frac{1}{16} \eta^6 - \frac{5}{128} \eta^8 + \ldots \right] \]
\[ = mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p^2}{m^2 c^2} \right) - \frac{1}{8} \left( \frac{p^2}{m^2 c^2} \right)^2 + \frac{1}{16} \left( \frac{p^2}{m^2 c^2} \right)^3 - \frac{5}{128} \left( \frac{p^2}{m^2 c^2} \right)^4 + \ldots \right] \]
\[ = mc^2 + \frac{p^2}{2m} + \ldots \]
\[ E = \gamma mc^2 \]
\[ = \frac{1}{\sqrt{1 - \beta^2}} mc^2 \]
\[ = mc^2 \left[ 1 + \frac{1}{2} \beta^2 + \frac{3}{8} \beta^4 + \frac{5}{16} \beta^6 + \frac{35}{128} \beta^8 + \ldots \right] \]
\[ = mc^2 + \frac{1}{2} m v^2 + \frac{3}{8} \frac{v^4}{c^2} + \frac{5}{16} \frac{v^6}{c^4} + \frac{35}{128} \frac{v^8}{c^6} + \ldots \]
\[ = mc^2 + \frac{1}{2} m v^2 + \ldots \]

Note that
\[ \frac{p^2}{2m} \neq \frac{1}{2} m v^2 \]
since \( p = \gamma m v \) but that they are approximately equal when \( v \ll c \).
4.1 Energy has Mass-Content

Actually, this is Photon-Energy has Mass Content. (Why equate photon-energy with “energy” or “pure energy”?)

Maybe I should use the relativistic momentum expression.

I should also probably provide Einstein’s argument (in a clear presentation) for historical purposes.

**Theorem.** Given that momentum of an energetically isolated system is conserved (Newton’s Second Law) and that electromagnetic radiation behaves as a photon (particle) with energy $E$ and momentum $p = E/c$, where $c$ is the relativistic speed limit, it can transfer (and carries) mass $m$:

$$m = \frac{E}{c^2}.$$  

Note that, in other senses, the photon has no mass. (This is an issue that should be cleared up, and appropriate distinguishing vocabulary should be developed.)

**Proof.** Max Born’s thought experiment: Imagine an energetically isolated train car (so that no external forces are present). Actually, this can just be a box of some mass $M$ and length $L$ isolated in deep space. The center of mass (COM) of such a system does not accelerate in an inertial frame and is stationary (in the COM frame). Let’s say a photon is expelled from an atom in the wall of the box on one side, and the photon flies to the other side of the box and is absorbed by an atom there a time $\Delta t$ later. While the photon is in transit, let’s say the mass of the box is $m_0$. With knowledge of the Compton effect (and other effects), we know that the photon will cause the box to recoil at a velocity $V$ in the opposite direction of the photon’s flight, and when the photon is absorbed, the box will experience an equal but oppositely directed impulse and come to rest, having moved a distance $d$. Since the center of mass has not moved but the box has, its mass must have been redistributed. Thus the photon has transferred mass (call it $m$) from one side of the box to the other such that the center of mass is stationary (and $M = m_0 + m$).

Figure 3: The photon’s transit
\begin{align*}
\Rightarrow \quad \frac{x^\text{initial}_{\text{COM}}}{m_0 + \dot{\vec{m}}} + \vec{m}(Q) &= \frac{x^\text{final}_{\text{COM}}}{m_0 + \dot{\vec{m}}} \\
\Rightarrow \quad \frac{m(L - d)}{m(L - d)} &= m_0 \dot{d} \\
\Rightarrow \quad m &= \frac{m_0 \dot{d}}{L - d} \\
&= \frac{m_0 \left(\frac{LE}{m_0 c^2 + E}\right)}{L - \left(\frac{LE}{m_0 c^2 + E}\right)} \\
&= \frac{m_0 LE}{[L(m_0 c^2 + E) - LE]} \\
&= \frac{m_0 LE}{Lm_0 c^2} \\
&= \frac{E}{c^2} 
\end{align*}

(1) Lemma 1: The center of mass of an energetically isolated system is stationary (in the COM frame)

(2) Definition 1: Definition of center of mass

(3) Simplification 1: Cancelling common factor \(m_0 + m\), substrackting common term \(m_0 L\)

(4) Reëxpression 1: Dividing by \(L - d\)

(5) Lemma 2: Substituting expression for \(d\)

\begin{align*}
\Delta t &= \frac{(L - d)}{c} \\
&= \frac{d}{V} \\
\Rightarrow \quad LV - dV &= dc \\
\Rightarrow \quad (d)(c + V) &= LV \\
\Rightarrow \quad d &= \frac{LV}{c + V} \\
&= \frac{LE/m_0 c}{c + E/m_0 c} \\
&= \frac{LE}{m_0 c^2 + E}
\end{align*}

(L1) Definition 1: Definition of photon velocity

(L2) Definition 2: Definition of box velocity
(L3) Reëxpression 1: Cross multiplying

(L4) Reëxpression 2: Collecting terms with common factor $d$

(L5) Reëxpression 3: Dividing by $c + V$

(L6) Sublemma 1: Substituting expression for $V$

\[ m_0 V = p \quad \text{(S1)} \]
\[ = \frac{E}{c} \quad \text{(S2)} \]
\[ \Rightarrow V = \frac{E}{m_0 c} \quad \text{(S3)} \]

(S1) Postulate 1: Conservation of momentum

(S2) Subsublemma 1: Momentum of a photon

- Refer to the Compton effect (1923), Planck’s relation for black-body radiation (1901), Einstein’s relation for the photoelectric effect (1905), and de Broglie’s relation (1922), as well as observation of radiation pressure by Lebedew (1901), Nichols and Hull (1903), and Gerlach et al. (1923)

(S3) Reëxpression 1: Dividing by $m_0$

(L7) Simplification 1: Multiplying numerator and denominator by $m_0 c$

(6) Simplification 2: Multiplying numerator and denominator by $m_0 c^2 + E$

(7) Simplification 3: Cancelling additive inverses

(8) Simplification 4: Cancelling common factor $m_0 L$
4.2 Mass has Energy-Content

Use the inverse of the previous argument, claiming that a whole massive object could be radiated away in the form of photons.

5 General Relativity Transformations

5.1 Killing Vector Diffeomorphisms

\[
x^\mu \rightarrow x'^\mu = \delta^\mu_{\mu'} \left( x^\mu + V^\mu \left( x \right) \right)
\]
\[
g_{\mu\nu} \rightarrow g'_{\mu'\nu'} = \delta^\mu_{\mu'} \delta^\nu_{\nu'} (g_{\mu\nu} - \nabla_\mu V_\nu - \nabla_\nu V_\mu)
\]

OR

\[
x^\mu \rightarrow x'^\mu = x^\mu + V^\mu \left( x \right)
\]
\[
g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} - \nabla_\mu V_\nu - \nabla_\nu V_\mu
\]