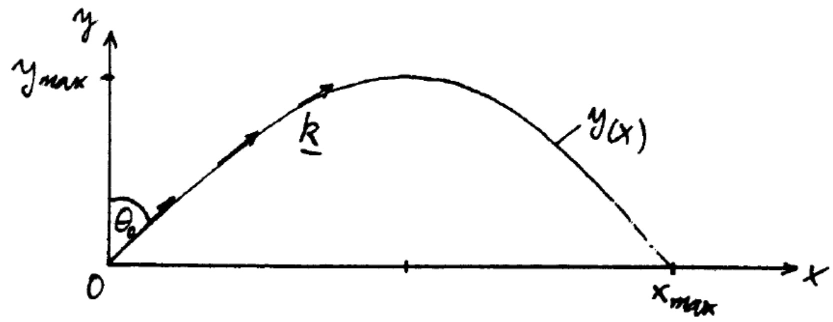
1. *Electricity and Magnetism* (Spring 1995, Part 1) [15 points]

Consider the propagation of electromagnetic waves in a non-uniform medium with refractive index $n^2 = 1 - {\omega_\mathrm{p}}^2/\omega^2$, where

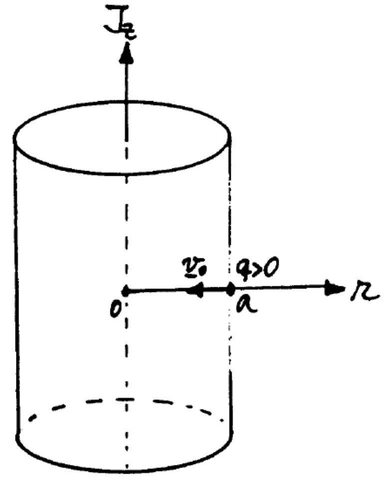
$$\frac{{\omega_\mathrm{p}}^2}{\omega^2} = \frac{y}{h} \qquad \omega, h = \text{ const.}$$

At the origin ($x = y = 0$) the plane wave propagates at an angle θ_0 with respect to the y axis. Using Snell's law, find the maximum height y_max that the wave assumes. Calculate the ray trajectory $y(x)$ and find the maximum horizontal distance x_max, where the wave reaches the original height ($y = 0$).



2. *Electricity and Magnetism* (Spring 1995, Part 1) [15 points]

A single charged particle ($q > 0$) is located at $r = a$ in the magnetic field of a long straight cylinder carrying a uniform axial DC current density J_z. At $t = 0$, the particle moves with initial velocity v_0 radially toward the axis. Sketch the particle's trajectory. Find the distance r_{min} of closest approach from the axis by integrating the equation of motion. For which v_0 does the particle reach the axis? Find the maximum distance r_{max} from the axis.



3. *Electricity and Magnetism* (Spring 1995, Part 1) [15 points]

A cavity resonator consists of an empty cube of sides $a = (1\text{ cm})$. What is the lowest resonance frequency? For the fundamental mode the electric field strength at $x = \frac{a}{4}$, $y = 0$, $z = \frac{a}{4}$ is found to be (0.5 V/cm). What is the direction and magnitude of the magnetic field at $x = \frac{3a}{4}$, $y = a$, $z = \frac{3a}{4}$?

4. *Electricity and Magnetism* (Spring 1995, Part 1) [20 points]

A circular loop of wire of radius R lies in the x-y plane centered at the origin. The wire carries a current $I(t)$ given by

$$I(t) = I_0 \cos \omega t$$

in a sense that is clockwise when viewed in the direction of the positive z axis. The frequency ω satisfies

$$\frac{\omega}{c} R \ll 1.$$

(a) Find the electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the region where

$$\frac{\omega}{c} R \ll \frac{\omega}{c} r \ll 1.$$

(b) Find the time average power radiated per solid angle $\frac{\mathrm{d}P}{\mathrm{d}\Omega}$ as a function of the polar angle θ and azimuthal angle ϕ.

5. *Electricity and Magnetism* (Spring 1995, Part 1) [20 points]

A spherical cavity of radius R centered at the origin is surrounded by an infinite medium of constant dielectric constant ε. A point charge q resides in the cavity along the z axis a distance $a < R$ from the center of the cavity.

(a) Find the electric field **E** and the displacement vector **D** outside the cavity as a power series in Legendre polynomials.

(b) Find the power series for the induced surface charge density σ on the spherical surface of the dielectric.

Hint:

$$\frac{1}{|\mathbf{r} - \mathbf{a}|} = \sum_{\ell} \frac{{r_<}^{\ell}}{{r_>}^{\ell+1}} P_\ell(\cos\theta)$$

6. *Electricity and Magnetism* (Spring 1995, Part 1) [20 points]

An infinitely long straight wire of negligible cross-sectional area carries a charge per unit length q_0 and is at rest along the z axis in an inertial frame K_0. The wire moves with velocity v directed along the positive z axis in the laboratory inertial frame K.

(a) Write down the electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{B}_0$ in the rest frame K. Lorentz transform the fields $\mathbf{E}_0$ and $\mathbf{B}_0$ to obtain the $\mathbf{E}$ and $\mathbf{B}$ in the laboratory frame K.

(b) By Lorentz transforming the charge and current densities in the rest frame, find the charge and current densities in the laboratory frame K.

(c) From the laboratory charge and current densities, determine directly the $\mathbf{E}$ and $\mathbf{B}$ fields in the laboratory.

Hint: If x, y, z, and t are the coordinates of a space-time point in the coordinate system K, the coordinates x', y', z', t' of the point in a coordinate system moving with velocity $v = v\hat{\mathbf{z}}$ relative to the first are

$$\begin{aligned} t' &= \gamma(t - \beta z) \\ z' &= \gamma(z - \beta t) \\ y' &= y \\ x' &= x \end{aligned} \qquad \text{where} \qquad \begin{aligned} \beta &= \frac{v}{c} \\ \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

7. *Statistical Mechanics and Thermodynamics* (Spring 1995, Part 1) [15 points]

Consider a gas of massive, nonrelativistic and noninteracting bosons in two dimensions. Show that this ideal Bose gas will not undergo Bose-Einstein condensation at any non-zero temperature.

8. *Statistical Mechanics and Thermodynamics* (Spring 1995, Part 1) [20 points]

The energy of an electric dipole $\mathbf{p}$ in a field $\mathbf{E}$ can be written $u = -\mathbf{p} \cdot \mathbf{E}$.

(a) Given a system of moments $\mathbf{p}$ that are non-interacting, find the average energy at temperature T.

(b) Show that the high-temperature polarization is given by the Curie law

$$P = \frac{p^2}{3k_{\mathrm{B}}T}E.$$

(c) Sketch the function $P(E)$.

1. *Quantum Mechanics* (Spring 1995, Part 2) [20 points]

Given a degenerate free electron gas with applied magnetic field $\mathbf{B} = B\hat{\mathbf{z}}\ldots$

(a) Choose the Landau gauge ($\mathbf{A} = \langle 0, B_x, 0 \rangle$). The solution to the Schrödinger equation is

$$\psi(x, y, z) = \exp\left[i(\alpha y + k_z z)\right] u(x).$$

Show that the eigenvalues must be of the form

$$\mathcal{E} = \left(u + \frac{1}{2}\right) \hbar\omega_c + \frac{\hbar^2}{2m} k_z.$$

What is ω_c?

(b) What is the degeneracy of levels for each u?

2. *Quantum Mechanics* (Spring 1995, Part 2) [15 points]

A single charged particle of charge e bound in a one-dimensional harmonic well is subjected to a small, uniform electric field of strength E. Let the particle mass be m, and oscillation frequency in the harmonic well be ω_0.

(a) Compute the shift in the ground-state energy to the lowest non-vanishing order in perturbation theory.

(b) Compute the exact shift in the ground-state energy.

(c) Compute the polarizability of the atom.

3. *Quantum Mechanics* (Spring 1995, Part 2) [15 points]

Consider an isotropic three-dimensional harmonic oscillator.

(a) Compute the degeneracy of the three lowest energy eigenstates.

(b) In each case, these states can be arrangd in appropriate linear combinations so that they are simultaneously energy eigenstates as well as eigenstates of the total angular momentum operator, L^2, and the z component of angular momentum, L_z. What are the allowed values of these quantum numbers that occur for each level and how many linearly independent states have each of these quantum numbers?

(c) Consider the effect of a small perturbation of the form

$$H' = BL_z.$$

What degeneracies of the unperturbed problem are lifted, and to what order of perturbation theory would it be necessary to go in order to first see this occur?

4. *Quantum Mechanics* (Spring 1995, Part 2) [20 points]

A possible test for time reversal and parity symmetry breaking is to look for evidence of an intrinsic electric dipole moment of the electron, analogous to its intrinsic magnetic dipole moment.

(a) Even if we assume that time reversal and parity are not symmetries of nature, it is possible to prove on the basis of the remaining symmetries of nature that the electric dipole moment must be parallel (or anti-parallel) to the electron spin. On the basis of what symmetry is this proof possible?

(b) Show that if either time reversal or parity are symmetries of nature that the electric dipole moment of the electron must be zero.

(c) If we assume that the electron has a non-zero dipole moment, $\boldsymbol{\mu}_{\text{el}}$, then there should be an added term in the Hamiltonian of the electron in the hydrogen atom of the form

$$H' = -\boldsymbol{\mu}_{\text{el}} \cdot \mathbf{E}(\mathbf{r})$$

where $\mathbf{E}(\mathbf{r})$ is the elctric field and $\mathbf{r}$ is the electron coordinate. This perturbation would lead to a small splitting between the 2S and 2P levels of the hydrogen atom. (Ignore, for present purposes, the fine structure splitting between these levels.) Would this splitting be first or second order in $\boldsymbol{\mu}_{\text{el}}$? In terms of $\boldsymbol{\mu}_{\text{el}}$ and properties of the hydrogen atom, estimate the size of this effect.

5. *Quantum Mechanics* (Spring 1995, Part 2) [20 points]

A particle of mass m and charge e is in a harmonic oscillator potential $V = k(x^2 + y^2 + z^2)/2$. At time $t = -\infty$ the system is in its ground state. It is then perturbed by a spatially uniform but time dependent electric field

$$\mathbf{E}(t) = A \exp\left[-(t/\tau)^2\right] \hat{\mathbf{z}},$$

where A and τ are constants. Calculate to lowest order the probability that the oscillator is in an excited state at $t = +\infty$.

6. *Quantum – Atomic Physics* (Spring 1995, Part 2) [20 points]

(a) An excited atom ($Z = 2$) is in an energy level with one electron occupying an S-state, i.e., $\ell = 0$, and the other a D-state, i.e., $\ell = 2$. The two electron spins are paired (antiparallel), so their total spin $S = 0$.

   (i) What is the smallest principal quanum number n for which this situation is possible?

   (ii) What is the atom's magnetic moment?

   (iii) The atom is placed in a small magnetic field $\mathbf{B}$ pointing along the z axis. Sketch the resulting splitting of the original energy level.

(b) Assume now that the S-state electron is excited to a P-state, i.e., $\ell = 1$, while the other electron remains in a D-state and the electron spins remain antiparallel.

   (i) What are the possible values of the atom's total orbital angular momentum quantum number L?

   (ii) Assume that the atom actually is in the $L = 3$ state. Will this state be split by the spin-orbit interaction?

(c) (i) Give the ground state electron configuration for nitrogen N ($Z = 7$).

   (ii) Give the ground state S, L, and J quantum numbers.

7. *Statistical Mechanics and Thermodynamics* (Spring 1995, Part 2) [15 points]

By heating a metal we produce thermal electrons. Recall that a dilute electron gas can be approximated as a monoatomic ideal gas with free energy given by

$$F(T, V, N) = N k_{\mathrm{B}} T \left(\ln \frac{N}{V} - \frac{3}{2} \ln T - c \right),$$

where c is a constant. Inside the metal, the free energy of the electrons can be approximated by

$$F(T, V, N) = -N\phi,$$

where ϕ is the work function. We ignore the dependence of the work function on T, $\frac{N}{V}$.

Derive the expression for the vapor pressure $p(T)$ of the electron gas in thermal equilibrium with the metal at temperature T, and discuss its qualitative behavior at high and low T.

8. *Statistical Mechanics and Thermodynamics* (Spring 1995, Part 2) [20 points]

As a simple statistical model of a rubber band, we take a one-dimensional chain consisting of N molecules. Each molecule can take two forms α, β. Let a, b be the lengths of each molecule, with $a \geq b$. Let N_α, $N_\beta = N - N_\alpha$ be the number of molecules of each type so that the total length L is given by

$$L(N_\alpha, N_\beta) = N_\alpha a + N_\beta b.$$

In the presence of tension X, the energy of a type α molecule is $\epsilon_\alpha - Xa$, and the energy of a type β molecule is $\epsilon_\beta - Xb$. Here, ϵ_α, ϵ_β are independent of temperature, and we take

$$\epsilon_\alpha > \epsilon_\beta.$$

(a) Calculate the partition function for the rubber at temperature T and tension X.

(b) Calculate the average length $\langle L \rangle$ at temperature T and tension X. Discuss its temperature dependence at zero tension.

(c) Calculate the elastic constant $K(T) \equiv \left(\frac{\partial X}{\partial Z}\right)_T$ at $X = 0$. Discuss its temperature dependence.