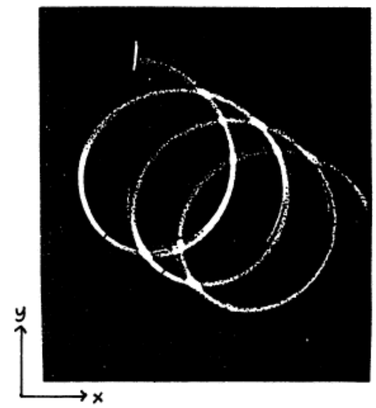
1. *Electricity and Magnetism* (Fall 1995, Part 1) [10 points]

The observed trajectory of an electron beam in a uniform electric and magnetic field is shown below. The beam energy is $\frac{1}{2}m{v_\perp}^2 = (100 \text{ eV})$ and the cyclotron radius is $r_0 = (2 \text{ cm})$. Find the direction and magnitude of **E** and **B**.

$$(e/m = 1.76 \times 10^{15}\,\text{cm}^2/\text{Vs}^2)$$



2. *Electricity and Magnetism* (Fall 1995, Part 1) [15 points]

Consider the skin effect in conductors.

(a) Using Ohm's law and Maxwell's equations, derive a differential equation for the current density $\mathbf{J}(r, t)$.

(b) Assume a straight cylindrical wire of radius a, uniform conductivity σ and permeability μ. For an AC current at a frequency ω, find the radial dependence of the current density. Sketch magnitude and phase versus radius.

(c) What is the scale length for the current penetration in iron ($\mu = 10^3 \mu_0$, $\sigma = 10^7\ \Omega^{-1}\mathrm{m}^{-1}$) at $f = (60\ \mathrm{Hz})$?

Hints:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$J_0\left(\sqrt{-i x}\right) = \mathrm{ber}(x) + i\,\mathrm{bei}(x)$$

3. *Electricity and Magnetism* (Fall 1995, Part 1) [15 points]

The capacitance between to confocal elliptical electrodes is given by

$$C = \frac{8\pi\epsilon_0\, c}{\ln\left(\dfrac{a_1 - c}{a_1 + c}\,\dfrac{a_2 + c}{a_2 - c}\right)}$$

where $c = \sqrt{{a_1}^2 - {b_1}^2} = \sqrt{{a_2}^2 - {b_2}^2}$ and a, b are semi-major and semi-minor axes, respectively. Derive the capacitance of the following related capacitors:

(a) Concentric spheres.

(b) Single prolate ellipsoid ($a > b$) with respect to a sphere of radius $R \to \infty$.

(c) Single elliptical rod ($a \gg b$).

(d) Flat ellipsoidal disk ($a \ll b$).

(e) Single electrode consisting of half an ellipsoidal ($a > b$) and half a sphere ($r = b$).

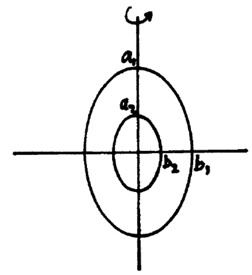


Figure 1: Confocal electrodes

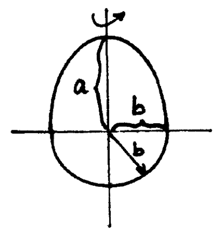
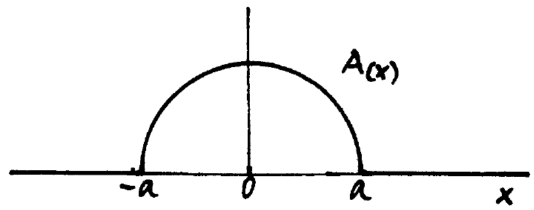


Figure 2: Half-ellipsoid / half-sphere

4. *Electricity and Magnetism* (Fall 1995, Part 1) [10 points]

An antenna has a distributed aperture function given by

$$A_{(x)} = \begin{cases} A_0\sqrt{1-(x/a)^2} & \text{for } x \le a, \\ 0 & \text{for } x \ge a. \end{cases}$$



Find the far-field radiation pattern $A_{(\theta)}$ by Fourier transformation,

$$A_{(\theta)} = \int_{-\infty}^{\infty} A_{(x)}\, e^{ikx\sin\theta}\, \mathrm{d}x.$$

Plot $A_{(\theta)}$ versus θ and find the angle for the first field null for $ka = 10\, z_{11} = 38$.

Hints:

$$J_{n(x)} = \frac{1}{i^n \pi}\int_0^\pi \cos(n\phi)\, e^{iz\cos\phi}\, \mathrm{d}\phi \qquad J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z}\, J_{n(z)}$$

5. *Electricity and Magnetism* (Fall 1995, Part 1) [10 points]

A dielectric separates two large parallel conducting plates of area A. The separation between them is t, and the dielectric constant ε varies linearly with position as

$$\varepsilon(x) = 1 + kx \qquad 0 \leq x \leq t.$$

A charge $+Q$ is placed on one plate and $-Q$ on the other.

(a) Find the electric displacement $D(x)$ $(0 \leq x \leq t)$.

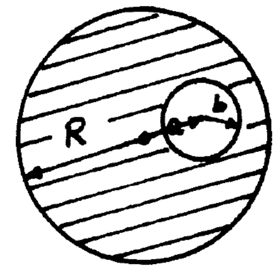
(b) Find the electric field $E(x)$ $(0 \leq x \leq t)$.

(c) Find the voltage V between the plates.

(d) Find the electrostatic force F between the plates.

6. *Electricity and Magnetism* (Fall 1995, Part 1) [10 points]

A long straight conductor carries a current I. As indicated in the figure, the conductor is in the form of a cylinder of radius R with an off-axis (by distance a) cylindrical hole of radius b. Find the magnetic field $\mathbf{H}$ in the hole, assuming the current density in the conductor is uniform.



7. *Electricity and Magnetism* (Fall 1995, Part 1) [15 points]

Consider a cubical cavity existing in the region

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1.$$

The "medium" in the cavity is vacuum.

(a) If the walls of the cavity are perfect conductors, determine the electric and magnetic fields (to within an overall scale and phase) corresponding to the lowest frequency mode with $E_y = E_z = 0$.

(b) Find the time average electromagnetic energy in the cavity corresponding to the mode in part (a).

(c) If the walls of the cavity have large, but finite, conductivity σ, magnetic permeability μ, and dielectric constant ε, obtain the expression for the Q of the cavity corresponding to the (approximate) mode in part (a).

Hints:

1. The Q of a cavity is $Q = \omega \dfrac{\text{Average energy stored}}{\text{Average power loss}}$

2. For a good conductor, the spatial dependence of the fields within the walls has the form $e^{-1-i/\delta\xi}$, where ξ measures the distance into the walls, and δ is the skin depth.

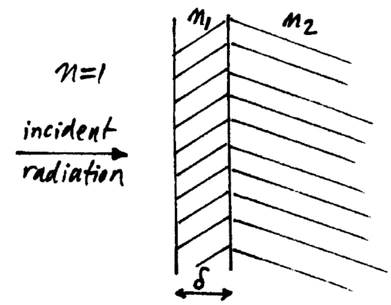
$$\delta = \frac{c}{\sqrt{2\pi\omega\mu\sigma}}$$

8. *Electricity and Magnetism* (Fall 1995, Part 1) [15 points]

Radiation of angular frequency ω is normally incident upon a semi-infinite slab of index of refraction n_2, which has been coated with a thin layer of material with index of refraction n_1. The thickness of this layer is δ. The magnetic permeabilities of both the slab and the thin layer have their vacuum value.

(a) Find a general expression for the ratio of reflected to incident intensities as a function of ω, n_1, n_2, and δ.

(b) Given n_2, find a set of values for n_1 and δ for which the reflected intensity at frequency ω is zero.



9. *Statistical Mechanics and Thermodynamics* (Fall 1995, Part 1) [? points]

The equation linking the magnetization, M, of a set of N magnetic moments and the magnitude of the applied field, H, is

$$M = \mu N \frac{H}{kT\sqrt{1 + (H/m_0\, kT)^2}}$$

where the quantities μ and m_0 are constants characteristic to the set of moments. When $H = M = 0$, the heat capacity of the set of moments is given by $C = Nc_0T^4$. Again, c_0 is a physical constant.

(a) What is the Gibbs magnetic energy, $G(N, T, H)$, of the system?

(b) Does this system violate the third law of thermodynamics? Justify your answer with a quantitative calculation.

Hints:

$$\mathrm{d}U = T\mathrm{d}S + H\mathrm{d}M + m\mathrm{d}N + \text{whatever else};$$

$$\int^x \frac{\mathrm{d}y}{\sqrt{1+y}} = 2\sqrt{1+x} + \text{constant}$$

10. *General Physics* (Fall 1995, Part 1) [15 points]

(a) Consider the reaction

$$p + p \rightarrow \bar{p} + p + p + p$$

in which two protons collide to form an antiproton and three protons. If one of the initial protons is at rest, find the *minimum* energy required of the other initial proton for the reaction to occur. Give your answer in terms of Mc^2, where M = proton mass.

(b) A Λ^0 particle lives for a time of (2.4×10^{-10} s) in its own rest system. If a Λ^0 particle is created in the laboratory with a velocity $v = \frac{1}{2}c$, how far does it travel before decaying? (c = velocity of light)

1. *Statistical Mechanics and Thermodynamics* (Fall 1995, Part 2) [10 points]

Suppose there are n atoms of an ideal gas per unit volume at temperature T, with mean free path $l = v\tau$ (v is the mean thermal velocity and τ is the average time e between collisions).

(a) Suppose an ideal gas is confined to a container with a small temperature difference between the ends, so that heat flows from one end to the other. Find an expression for the thermal conductivity of the gas, in terms of the specific heat and the average temperature of the gas. The thermal conductivity is the coefficient relating the temperature difference to the heat flow. You can approximate the average velocity as constant over length scales of the order of the mean free path.

(b) Explain why the thermal conductivity is independent of pressure (over a wide range of particle densities) at constant temperature.

2. *Statistical Mechanics and Thermodynamics* (Fall 1995, Part 2) [10 points]

(a) Using the grand canonical distribution, derive the following expression for the particle number fluctuation:

$$\left\langle (N - \langle N \rangle)^2 \right\rangle = k_{\mathrm{B}} T \left(\frac{\partial \langle N \rangle}{\partial \mu} \right)_{T,V}.$$

(b) Consider a substance that has the equation of state of the form

$$PV = \phi(T)$$

where $\phi(T)$ is some function of T, and with energy E a function of T only, i.e.,

$$E = \Psi(T).$$

By making use of the first and second laws of thermodynamics, show that the substance must actually obey the ideal gas equation, i.e.,

$$PV = RT$$

with R a constant.

3. *Statistical Mechanics and Thermodynamics* (Fall 1995, Part 2) [10 points]

A very crude model of an enzyme (e.g., myoglobin or hemoglobin) consists of a system of n binding sites for binding molecules of a substrate substance (e.g., oxygen). We associate with each site a variable μ_i, $(i = 1, 2, \ldots, n)$ which takes the value $+1$ (-1) if the i-th site is occupied (unoccupied) by a substrate molecule. A configuration of the enzyme molecule is then specified by the set of values $(\mu_1, \mu_2, \ldots, \mu_n)$, and will be denoted by $\{\mu\}$. For a particular configuration $\{\mu\}$ then, the number of occupied sites is given by

$$\sum_{i=1}^{m} \frac{1}{2}(1 + \mu_i).$$

(a) Assume that the binding sites are independent (non-interacting), and let

$$p(+1) = Ce^{-J}, \qquad p(-1) = Ce^{J},$$

give the probability that a site is occupied $(+1)$ or unoccupied (-1), respectively.

(i) What requirement determines C for given J?

(ii) Find the average number of occupied sites N.

(b) Include now the effects of interactions among binding sites. Assume that only nearest-neighbor sites interact, and take the occupation probability distribution to be of the form

$$P(\{\mu\}) = Z^{-1} \prod_{i=1}^{n} \exp(-J\mu_i) \exp(U\mu_i \, \mu_{i+1}).$$

(i) What requirement determines Z?

(ii) Show that the average number of occupied sites is given by

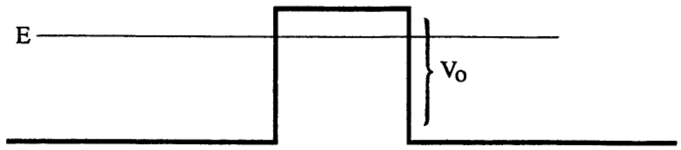
$$N = \frac{n}{2} + \frac{1}{2}M$$

where M is the "magnetization" defined by

$$M = -\frac{\partial}{\partial J}(\ln Z).$$

4. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

Consider the one-dimensional square potential barrier below.



When the transmitted wave function amplitude ratio T is very small comparted to the incident amplitude, T is approximately given by

$$T = \exp\left\{ -\frac{\left[2m(v_0 - E)\right]^{1/2}}{\hbar} \right\}.$$

(a) Suppose that $V(x)$ is a slowly varying potential. Apply the result above to show that

$$|T|^2 \approx \exp\left\{ -2\int_{x_1}^{x_2} \mathrm{d}x \, \frac{\left[2m(v_0 - E)\right]^{1/2}}{\hbar} \right\}.$$

(b) Now suppose that a metallic object is placed in an electric field, so that near the surface the potential seen by the highest energy electron varies as

$$\begin{aligned} V(x) &= 0 && (x < 0), \\ V(x) &= V_0 - eEx && (x > a). \end{aligned}$$

The difference $V_0 - E$ represents the work function. The electron sees a triangular potential barrier extending from $x = 0$ to $x = a$, where a is defined by

$$E = V(L) = V_0 - eEa,$$

and therefore

$$L = (V_0 - E)/e = W/eE.$$

An electron of energy E then has a certain probability of penetrating this barrier. Show that the ratio T is given approximately by

$$|T| \approx \exp(-E_0/E),$$

where E_0 is the characteristic field for field emission of electrons. Make an estimate of the characteristic field (i.e., put in some reasonable numbers).

5. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

Stark effect of $n = 2$ states for the hydrogen atom: In this problem, an electric field partially lifts the degeneracy of the electronic eigenvalues. The perturbing Hamiltonian is given by

$$H_1 = eEz.$$

The unperturbed wave functions are given by $\psi_{n\ell m}$, where

$$\begin{aligned} \psi_{200} &= (2a_0)^{-3/2} 2(1 - r/2a_0) e^{-r/2a_0} Y_{00} & \quad Y_{00} &= \tfrac{1}{\sqrt{4\pi}} \\ \psi_{211} &= (2a_0)^{-3/2} \tfrac{1}{\sqrt{3}} \left(\tfrac{\mu}{a_0}\right) e^{-r/2a_0} Y_{11} & \quad Y_{10} &= \sqrt{\tfrac{3}{4\pi}} \cos\theta \end{aligned}$$

(a) Write down the equations for the corrections to the ground state energy that occur with the application of the perturbing field.

(b) Evaluate the corrections and make an energy level diagram. Label the levels with the new eigenfunctions (in terms of the unperturbed functions above).

6. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

A particle of mass m moves in a one-dimensional potential of the form

$$V(x) = V_0 \left(|x| / a \right)^{\alpha}$$

with $\alpha > 0$.

(a) For $\alpha = 2$, this is the harmonic oscillator problem. It is easy to solve this problem exactly. Do not do so, except as a check on your methods (if you wish to). Instead, use the uncertainty principle to estimate the ground-state energy and the spatial extent of the ground-state.

(b) Now repeat the same calculation as in part (a), but for $\alpha = 1$ and $\alpha = 4$.

(c) There is a dimensionless parameter, $y = h^2/2ma^2V_0$, in this problem. For $y \gg 1$, the ground-state energy is an increasing function of α while for $y \ll 1$, it is a decreasing function of α. Explain physically why this is so.

7. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

Consider as a model of the H^2 molecule, a pair of protons held a fixed distance R apart by a rigid bar which represents the effects of the electrons. Each proton has mass M and spin $\frac{1}{2}$. Ignore all effects of spin-orbit coupling. Therefore, the spins can be arranged into a state of total spin 1 or of total spin 0.

(a) Show that the ground state has spin 0 and compute the energy splitting between the spin 0 and the spin 1 ground states in units of $e = h^2/2MR^2$.

(b) If the proton were a spin $\frac{1}{2}$ boson (instead of a fermion), how would the results in part (a) be altered?

8. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

A hydrogen atom is a bound state of an electron and proton.

(a) Ignoring all relativistic effects, such as spin-orbit coupling, hyperfine coupling, etc., what is the degeneracy of the first excited state of the hydrogen atom?

(b) Some of these degeneracies are accidental, i.e., due to the fact that a number of constants vanish in the limit $c \to \infty$, and not due to any symmetry of nature. When these other weak couplings are considered, the same states should be classified only in terms of the true symmetries of nature, such as especially full rotational symmetry, i.e., the states should be labeled by their total angular momentum, J. When the effect of the additional very weak relativistic effects are considered, the highly degenerate excited state in part (a) should be split into several sets of states with only the degeneracies implied by symmetry. Determine how many distinct energies will result from this slitting, and what the remaining degeneracies of those states will be. Specify also at least the total angular momentum of each of these staes. You do not need to specify the form of the perturbing Hamiltonian, nor determine which of these states have lower or higher energies than the others.

9. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

As the simplest possible model of an atom, we consider an electron of mass m, confined in a harmonic potential of the form $V(r) = \frac{1}{2}Kr^2$. Compute the polarizability of this atom.

Hint: The polarizability is related to the change in the energy of a system when subjected to an electric field to second order in the field strength according to

$$\text{Change in energy} = -\frac{1}{2}\alpha E^2.$$

10. *Quantum Mechanics* (Fall 1995, Part 2) [10 points]

A particle of charge e and mass m moves in constant perpendicular E and B fields:

$$\mathbf{E} = \langle 0, 0, E \rangle \qquad \mathbf{B} = \langle 0, B, 0 \rangle$$

(a) Write down the Schrödinger equation.

(b) Show that, by separation of variables, this reduces to a 1-dimensional problem of a simple harmonic oscillator.

(c) Find the expectation value of the velocity in the x direction:

$$v_x = \mathrm{d}x/\mathrm{d}t = \frac{1}{i\hbar}[x, H]$$

(in any energy eigenstate).

11. *Quantum Mechanics* (Fall 1995, Part 2) [? points]

Two spin-1/2 nuclei fixed in a crystal lattice are subject to a uniform external magnetic field of magnitude B parallel to the z-axis. Each nucleus has an intrinsic magnetic moment represented by the operators $K\boldsymbol{\sigma}^{(1)}$ and $K\boldsymbol{\sigma}^{(2)}$, respectively, where K is a constant.

(a) Neglecting the magnetic dipole interactions between the nuclei, the Hamiltonian of the system is

$$H_1 = H_0 - KB\left(\sigma_z^{(1)} + \sigma_z^{(2)}\right)$$

where H_0 is independent of spin. Let ψ be the spin-independent eigenfunction corresponding to a given nondegenerate eigenvalue E_0 of H_0, and suppose the nuclei are in a triplet spin state. Find the corresponding eigenvectors and eigenvalues of H_1.

(b) Include now the magnetic dipole interaction between the nuclei given by

$$V = \frac{K^2}{r^3}\left[\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} - 3\left\{\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{r}}\right\}\left\{\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{r}}\right\}\right].$$

Here $\mathbf{r}$ is the position vector from the nucleus 1 to nucleus 2, and $\hat{\mathbf{r}}$ the unit vector in the direction of $\mathbf{r}$. Thus the total Hamiltonian of the system now becomes $H = H_1 + V$. To first order in perturbation theory, compute the shift in the energy levels found in part (a) due to the presence of V.