1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a \cdot \hat{\mathbf{n}}_a \, \mathbf{s}_b \cdot \hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

2. *Quantum Mechanics* (Fall 2004)

The van der Waals interaction between two neutral atoms is due to dipole-dipole interactions. Consider the following simplified 1-D model. Each atom consists of a fixed nucleus of charge $+e$ and electron of charge $-e$, bound by a harmonic spring. Two such oscillators are a distance R ($\gg$ size of the atom) apart. The Hamiltonian of the system consists of the two harmonic oscillator terms plus a dipole-dipole perturbation.

(a) Write the perturbation part of the Hamiltonian.

(b) Calculate the correction to the energy of the unperturbed ground state. This is the van der Waals interaction potential. (*Hint*: it should come out $\propto 1/R^6$.)

3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\left\langle \mathbf{r}_+, \mathbf{r}_- \middle| \mathbf{r}'_+, \mathbf{r}'_- \right\rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\,\delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\,|\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

[1]Write your answer in terms of m, e^2 or α, $\hbar$, c, the Bohr radius, etc. You may use units in which $\hbar = c = 1$.

4. *Quantum Mechanics* (Fall 2004)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally, the states are labeled $|n, l, j, m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

(a) If the electron is in the state $|n, l, j, m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $\mathbf{J}_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_+ \middle| 3, 1, \frac{3}{2}, -\frac{1}{2} \right\rangle = ?$

(ii) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_+ \middle| 3, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(iii) $\left\langle 2, 1, \frac{3}{2}, \frac{3}{2} \middle| p_z \middle| 2, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(iv) $\left\langle 2, 1, \frac{1}{2}, -\frac{1}{2} \middle| \mathbf{L}^2 \middle| 2, 1, \frac{1}{2}, -\frac{1}{2} \right\rangle = ?$

(v) $\left\langle 3, 2, \frac{3}{2}, -\frac{1}{2} \middle| \mathbf{J}^2 \middle| 3, 2, \frac{3}{2}, -\frac{1}{2} \right\rangle = ?$

(vi) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_z \middle| 3, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(d) What is $\left\langle 1, 0, \frac{1}{2}, \frac{1}{2} \middle| p_i\, p_j \middle| 1, 0, \frac{1}{2}, \frac{1}{2} \right\rangle = ?$

(e) For given n, l, j, and m, what are the conditions on n', l', j', and m' so that

$$\left\langle n', l', j', m' \middle| \mathbf{s} \cdot \mathbf{r} \middle| n, l, j, m \right\rangle \neq 0?$$

5. *Quantum Mechanics* (Fall 2004)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, ...$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1| e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $\big|\psi(0)\big\rangle = |\phi\rangle$. What is $\big|\psi(t)\big\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

6. *Statistical Mechanics and Thermodynamics* (Fall 2004)

If the specific heat of a gas of non-interacting fermions in d dimensions varies with temperature as $C \sim T^{\alpha}$ for $k_B T \ll E_F$, then what is α? What is α for a system of non-interacting bosons?

7. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Some organic molecules have a triplet excited state at energy $k_B\Delta$ above a singlet ground state.

(a) Find an expression for the magnetic moment in a field B in terms of Δ, B, the temperature T, the Bohr magneton μ_B, and the gyromagnetic ratio g.

(b) Show that the susceptibility for $T \gg \Delta$ is given by $N(g\mu_B)^2/2k_BT$, where N is the total number of molecules in the system.

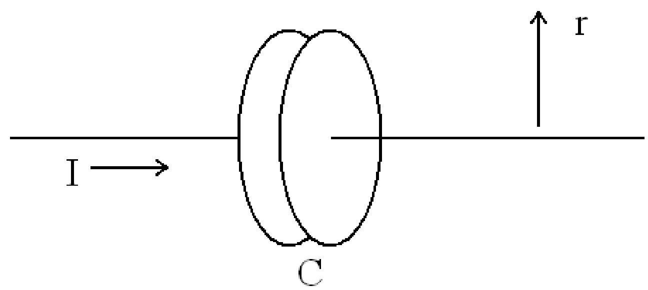
(c) With the help of a diagram of energy levels versus field and a rough sketch of entropy versus field, explain how this system might be cooled by adiabatic magnetization (*not demagnetization*).

8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?

9. *Electricity and Magnetism* (Fall 2004)

A wire carrying current I is connected to a circular capacitor of capacitance C, as depicted in the figure. What is the magnetic field outside the wire, far from the capacitor (as a function of the distance r from the wire)? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor. What is this magnetic field?



10. *Electricity and Magnetism* (Fall 2004)

The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

11. *Electricity and Magnetism* (Fall 2004)

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $\mathbf{r}(t)$. You do not need to find dimensionless proportionality constants (i.e., only find the dependence on q, $\mathbf{r}(t)$, and universal constants).

12. *Electricity and Magnetism* (Fall 2004)

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy E_{min} of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2} r M^2 + u M^4 - h M$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^{\delta}$. What is δ?

14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature T. What is the average energy per photon in units of kT?

You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$