1. *Quantum Mechanics* (Fall 2002)

A Stern-Gerlach apparatus is adjusted so that the z-component of the spin of an electron (spin-1/2) transmitted through it is $-\hbar/2$. A uniform magnetic field in the x-direction is then switched on at time $t = 0$.

(a) What are the probabilities associated with finding the different allowed values of the z-component of the spin after time T?

(b) What are the probabilities associated with finding the different allowed values of the x-component of the spin after time T?

2. *Quantum Mechanics* (Fall 2002)

The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2,$$

where the magnetic field $\mathbf{B}$ is related to the vector potential $\mathbf{A}$ by $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Here, e is the charge of the particle, m the mass, c the velocity of light and $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum of the particle. Let $\mathbf{A} = -B_0 y\hat{\mathbf{x}}$, corresponding to the magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.

(a) Find the energy levels of the particle.

(b) Would the energy levels change if we chose $\mathbf{A}$ to be $\frac{B_0}{2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$? Give reasons for your answer.

3. *Quantum Mechanics* (Fall 2002)

A charged particle of charge, q, and mass, m, is bound in a one-dimensional harmonic oscillator potential $V = \frac{1}{2}m\omega^2 x^2$, where ω is the frequency of the oscillator. The system is then placed in an electric field E that is constant in space and time.

(a) Calculate the shift of the ground state energy to order E^2.

(b) What are the third and higher order (in E) shifts in the ground state energy? Give reasons for your answer.

Hint: If n labels the eigenstates of the unperturbed harmonic oscillator, then
$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n'}\,\delta_{n,n'-1} + \sqrt{n'+1}\,\delta_{n,n'+1}\right]$.

4. *Quantum Mechanics* (Fall 2002)

Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r}$$

(a) What is the ground state energy of this Hamiltonian?

(b) What is the expectation value of the potential energy $\left\langle -\frac{Ze^2}{r} \right\rangle$ in the ground state?

(c) What is the expectation value of the kinetic energy $\left\langle -\frac{\hbar^2}{2m}\nabla^2 \right\rangle$ in the ground state?

5. *Quantum Mechanics* (Fall 2002)

In the Born-Oppenheimer approximation, the electrons are treated quantum mechanically, while the atomic nuclei are treated classically. The electronic energy is calculated as a function of the spacing between the nuclei. The sum of the electronic energy and the potential energy due to nuclei-nuclei interactions is minimized. The nuclear kinetic energy is neglected. As a toy model, we will consider the formation of a diatomic molecule in one dimension. Let us suppose that the electron is at $\mathbf{x}$ and the nuclei are at $\mathbf{X}_1$, $\mathbf{X}_2$. We assume that the interaction between and electron and a nucleus is $V(\mathbf{x}-\mathbf{X}_i) = -V_0\,\delta(\mathbf{x}-\mathbf{X}_i)$ for $i = 1, 2, V_0 > 0$. The interaction between nuclei is $U(\mathbf{X}_1 - \mathbf{X}_2) = \frac{Z^2e^2}{|\mathbf{X}_1-\mathbf{X}_2|}$.

(a) Suppose that the nuclei are a distance a apart. What is the ground state energy of an electron to order a if $\frac{mV_0 a}{\hbar^2} \ll 1$ (m is the electron mass).

(b) Consider a diatomic molecule composed of an electron and two nuclei. Using the Born-Oppenheimer approximation, find the separation between the two nuclei if $V_0 \gg Z^2e^2$. You need only compute the separation to lowest order in $Ze/V_0^{1/2}$.

6. *Statistical Mechanics and Thermodynamics* (Fall 2002)

A gas of N highly relativistic, and non-interacting, spin 1/2 Fermions occupies a volume V at a temperature that is effectively equal to zero.

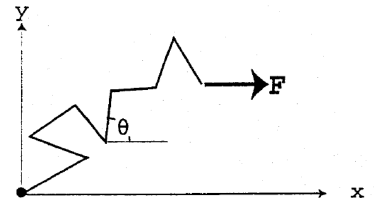
(a) Find the pressure on this gas.

(b) Based on the calculation you have just done, show what (extreme) inequality must be satisfied in order that the assumption of a temperature that is "effectively equal to zero" is justified.

(c) Suppose that the energy of the system due to gravitational self-attraction goes as $-AN^2V^{-1/3}$, where A is a constant. What does this and your result for the pressure imply about the stability of this system, assuming that gravitational attraction is what keeps it together?

7. *Statistical Mechanics and Thermodynamics* (Fall 2002)

A chain consists of N links that can freely rotate in two dimensions. The links are joined end-to-end, as shown below.



The chain is subjected to a tension, F, in the x-direction, as indicated. The tension is applied at the end of the chain, so that the total energy of the chain is given by

$$E = -Fl \sum_{i=1}^{N} \cos\theta_i$$

where θ_i is the angle that the i^{th} link makes with the x-axis, and l is the length of each link in the chain.

(a) Calculate the partition function of this chain.

(b) From the partition function, find the relationship between the extension of the chain in the x-direction and the tension, F, assuming that the temperature is T.

(c) When the tension, F, is small, the extension-versus-tension expression implies a spring constant for the freely jointed chain. What is this effective spring constant?

If the integrals do not evaluate to elementary functions in parts a and b, it is not necessary to attempt to reduce them. Leave them as integrals. However, in part c, it is necessary to come up with something explicit.

8. *Electricity and Magnetism* (Fall 2002)

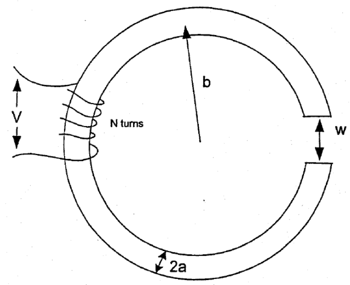
Radiating Charges

(a) A point charge q under acceleration $\mathbf{a}(t)$ emits electromagnetic radiation. Give qualitative physical arguments why the radiated power, P, should be of the form $P = Bq^2a^2$, where B is a proportionality constant. Determine by dimensional analysis the dependence of B on fundamental physical constants. Explain how and why the exact expression for B differs from this estimate.

(b) A point charge q has mass m and is attached to a spring (of spring constant κ) hanging form a fixed support above an infinite horizontal **conducting** plane. The charge is set in motion with amplitude $A < h$, the equilibrium height of the charge above the conducting plane. Calculate its instantaneous radiating power.

9. *Electricity and Magnetism* (Fall 2002)

A D.C. electromagnet is constructed from a cylindrical soft-iron bar with radius a. The relative magnetic permeability of the iron is μ. The bar is bent into a C-shape as shown below with radius b. The width of the small gap is w. The magnet is energized by winding a coil of copper wire N turns tightly around the bar and connecting the coil to a D.C. power supply with voltage V. The copper wire has resistivity ρ, and radius r_{wire}. Assume $r_{wire} \ll a \ll b$ and ignore fringe-field effects.



(a) What is the steady-state value of the magnetic field B in the gap?

(b) What is the time constant governing the response of the current in the coil when the voltage is turned on? (Assume μ is constant.)

10. *Electricity and Magnetism* (Fall 2002)

A point charge q is **inside** a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

(a) the potential inside the sphere;

(b) the induced surface-charge density at the point on the sphere nearest to q [Editor's Note: You may assume that the outer radius is different from the inner radius so the sphere is not an infinitesimal shell.];

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a **total** charge Q on its inner and outer surface?

11. *Electricity and Magnetism* (Fall 2002)

Describe how you would *measure* the following physical quantities:

(a) An electrostatic field $\mathbf{E}$.

(b) A vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}$ in the gauge $\nabla \cdot \mathbf{A} = 0$.

(c) The charge of an electron assuming its mass is known.

(d) The speed of light of electromagnetic waves.

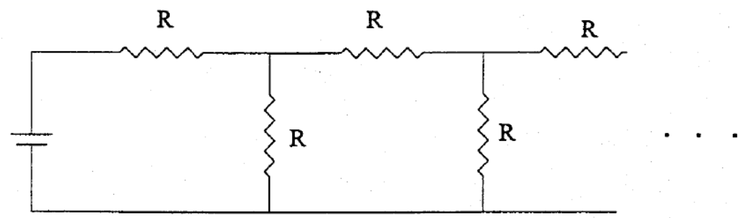
(e) The electrical conductivity of a flame.

(f) The direction of wave propagation of a plane electromagnetic wave.

Please describe the approach and method as realistically as possible.

12. *Electricity and Magnetism* (Fall 2002)

A voltage is applied to the infinitely long resistor network shown below. Each resistor has the same resistance R. Calculate the power dissipated in each resistor.



13. *Statistical Mechanics and Thermodynamics* (Fall 2002)

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperate T.

Find the a) chemical potential, b) free energy, c) entropy, d) pressure, and e) heat capacity at constant pressure.

14. *Statistical Mechanics and Thermodynamics* (Fall 2002)

In this problem, you will study the q-state Potts model using mean-field theory. The Hamiltonian is

$$H_{\text{Potts}} = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}$$

where the 'spins' σ take values $\sigma_i = 0, 1, 2, ..., q-1$. For $q = 2$, this is the Ising model.

(a) Show that H_{Potts} can be rewritten in the form

$$H_{\text{Potts}} = -\frac{J}{q} \sum_{\langle i,j \rangle} \left[(q-1) \mathbf{s}_i \cdot \mathbf{s}_j + 1 \right]$$

where the vectors $\mathbf{s}_i$ are constrained to take values in a set of q vectors in $(q-1)$-dimensional space, $\{\mathbf{S}_1, \mathbf{S}_2, ..., \mathbf{S}_q\}$ satisfying $\mathbf{S}_a \cdot \mathbf{S}_b = 1$ if $a = b$ and $\mathbf{S}_a \cdot \mathbf{S}_b = -\frac{1}{q-1}$ if $a \neq b$.

(b) In the mean-field theory, we approximate the Hamiltonian H_{Potts} by a mean-field Hamiltonian H_{MF},

$$H_{\text{MF}} = \sum_i \left[\mathbf{h} \cdot \mathbf{s}_i - \frac{J}{q} \right]$$

in which there is no interaction between the different spins, but each spin is coupled to an effective magnetic field, $\mathbf{h}$. Calculate the partition function of H_{MF}.

(c) Using H_{MF}, compute $\langle \mathbf{s} \rangle$. Impose the self-consistency condition that the effective magnetic field is generated by the average spin $\langle \mathbf{s} \rangle$ so that H_{MF} approximates H_{Potts}. Requiring self-consistency, derive (but do not solve) the mean-field equation for $\langle \mathbf{s} \rangle$. (For simplicity, you may assume that the tetrahedron is oriented so that one of the allowed values of $\mathbf{s}_i$ is $(0, 0, ..., 0, 1)$. Assume that $\langle \mathbf{s} \rangle = s(0, 0, ..., 0, 1)$ and find s.)