Summer Lecture Notes: Lecture 6

The Quantum Harmonic Oscillator

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Contents

1 Preliminaries

This is meant to be a quick review with a couple thoughtful questions and examples. In these notes I'm basically rewriting what Sakurai [1] has in section 2.3, except I improve on his notation. We'll use the following relations (letting $\mathrm{d}_x \equiv \frac{\mathrm{d}}{\mathrm{d}x}$):

$$[X, P] = i\hbar$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$\langle x| X = x \langle x| \qquad \langle x| P = -i\hbar\, \mathrm{d}_x \langle x| \tag{1}$$

$$\langle p| X = i\hbar\, \mathrm{d}_p \langle p| \qquad \langle p| P = p \langle p| \tag{2}$$

2 One 1-D Simple Quantum Harmonic Oscillator

2.1 Completing the Square

We start with the Schrödinger equation for the 1-D simple harmonic oscillator:

$$i\hbar\, \partial_t |\psi\rangle = \left(\frac{P^2}{2m} + \frac{m\omega^2 X^2}{2} \right) |\psi\rangle \qquad \omega = \sqrt{k/m}$$

The Hamiltonian is of a form that may be transformed by "completing the square" using complex numbers, which yeilds a convenient representation. Here's the basic transformation:

$$\begin{aligned} H &= (B^2 + A^2) \\ &= (A - iB)(A + iB) - i(AB - BA) \\ &= (A - iB)(A + iB) - i[A, B] \end{aligned}$$

Since the operators A and B contain X and P, $[A, B]$ gives us the commutator of X and P, which is a number (rather than another operator). To see where the square comes in, we define $C \equiv A + iB$, with $C^\dagger = A^\dagger - iB^\dagger = A - iB$ since A and B are Hermitean, via the Hermiticity of X and P. Thus we have reëxpressed the Hamiltonian as $H = C^\dagger C - i[A, B]$, which would be $H = |C|^2$ if C were a number.

2.2 Unmemorable Notation

Completing the square (the conventional way, with i accompanying P rather than X, and conveniently picking the form $(A - iB)(A + iB) - i[A, B]$ rather than $(A + iB)(A - iB) + i[A, B]$), we have

$$\begin{aligned} H &= \left(\frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}\right) \\ &= \hbar\omega\left(\frac{m\omega X^2}{2\hbar} + \frac{P^2}{2m\hbar\omega}\right) \\ &= \hbar\omega\left\{\left(\sqrt{\frac{m\omega}{2\hbar}}X - \frac{iP}{\sqrt{2m\hbar\omega}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}X + \frac{iP}{\sqrt{2m\hbar\omega}}\right) - \frac{i}{2\hbar}(XP - PX)\right\} \\ &= \hbar\omega\left\{a^\dagger a + \frac{1}{2i\hbar}(i\hbar)\right\} \\ &= \hbar\omega\left(N + \tfrac{1}{2}\right) \end{aligned}$$

where we've used a known commutator and the definitions of $a^\dagger$, a, and N (the creation, annihilation, and number operators):

$$(XP - PX) = [X, P] = i\hbar$$

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right) \qquad a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right)$$

$$N \equiv a^\dagger a$$

Note that

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \qquad P = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$$

2.3 Memorable Notation

Let's define

$$x_0 \equiv \sqrt{\frac{\hbar}{2m\omega}} \qquad p_0 \equiv \sqrt{\frac{\hbar m\omega}{2}}$$

which will define the length and momentum scales of the oscillator[1], and rewrite the preceding statements using these constants. (The expressions will be more memorable with this notation.) Note that

$$x_0 p_0 = \frac{\hbar}{2}$$

[1] In fact, we will find that x_0 and p_0 are the standard deviations of the ground state probability density in the position and momentum bases, respectively.

Completing the square (the conventional and convenient way)[2], we have

$$\begin{aligned}
H &= \left(\frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}\right) \\
&= \hbar\omega\left(\frac{m\omega X^2}{2\hbar} + \frac{P^2}{2m\hbar\omega}\right) \\
&= \hbar\omega\,\frac{1}{4}\left(\frac{X^2}{{x_0}^2} + \frac{P^2}{{p_0}^2}\right) \\
&= \hbar\omega\left\{\frac{1}{2}\left(\frac{X}{x_0} - i\frac{P}{p_0}\right)\frac{1}{2}\left(\frac{X}{x_0} + i\frac{P}{p_0}\right) - \frac{i}{4x_0p_0}(XP - PX)\right\} \\
&= \hbar\omega\left\{a^\dagger a - \frac{i}{2\hbar}(i\hbar)\right\} \\
&= \hbar\omega\left(N + \tfrac{1}{2}\right)
\end{aligned}$$

where we've used

$$(XP - PX) = [X, P] = i\hbar = i2x_0p_0$$

$$a^\dagger \equiv \frac{1}{2}\left(\frac{X}{x_0} - i\frac{P}{p_0}\right) \qquad a \equiv \frac{1}{2}\left(\frac{X}{x_0} + i\frac{P}{p_0}\right) \tag{3}$$

$$N \equiv a^\dagger a$$

Note that

$$X = x_0(a^\dagger + a) \qquad P = ip_0(a^\dagger - a) \tag{4}$$

2.4 Application of $a^\dagger$ and a (Energy Eigen-states and -values and the Null State)

It shall be useful to know a few mathematical relations:

$$N = a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2}$$

$$[a, a^\dagger] = 1$$

$$[N, a^\dagger] = a^\dagger$$

$$[N, a] = -a$$

Here are proofs of the previous equations:

$$\frac{H}{\hbar\omega} - \frac{1}{2} = \frac{\hbar\omega\left(N + \frac{1}{2}\right)}{\hbar\omega} - \frac{1}{2} = N = a^\dagger a$$

[2]Note that here and previously we have "extracted" $\hbar\omega$ from the expression, which has units of energy, and created a dimensionless expression inside the parentheses. The dimensionlessness is made more obvious with the presence of the scale factors x_0 and p_0.

$$\begin{aligned}
[a, a^\dagger] &= aa^\dagger - a^\dagger a \\
&= \frac{1}{4}\left(\frac{X}{x_0} + i\frac{P}{p_0}\right)\left(\frac{X}{x_0} - i\frac{P}{p_0}\right) - \frac{1}{4}\left(\frac{X}{x_0} - i\frac{P}{p_0}\right)\left(\frac{X}{x_0} + i\frac{P}{p_0}\right) \\
&= \left[\frac{1}{4}\left(\frac{X^2}{{x_0}^2} + \frac{P^2}{{p_0}^2}\right) - \frac{i}{4x_0p_0}[X, P]\right] - \left[\frac{1}{4}\left(\frac{X^2}{{x_0}^2} + \frac{P^2}{{p_0}^2}\right) + \frac{i}{4x_0p_0}[X, P]\right] \\
&= -\frac{i}{4x_0p_0}[X, P] - \frac{i}{4x_0p_0}[X, P] \\
&= -\frac{i}{2x_0p_0}(i2x_0p_0) \\
&= 1
\end{aligned}$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a = a^\dagger(1) + (0)a = a^\dagger$$

$$[N, a] = [a^\dagger a, a] = a^\dagger[a, a] + [a^\dagger, a]a = a^\dagger(0) - [a, a^\dagger]a = 0 - (1)a = -a$$

Now, we know that energy eigenstates of H must also be eigenstates of N, so we'll denote those (normalized) orthogonal eigenstates by $|n\rangle$, such that $N|n\rangle = n|n\rangle$ and $H|n\rangle = E_n|n\rangle$ with $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. The only constraint on n that we know right now is that it must be a real number. Since we have

$$\begin{aligned}
Na^\dagger|n\rangle &= ([N, a^\dagger] + a^\dagger N)|n\rangle \\
&= (a^\dagger + a^\dagger n)|n\rangle \\
&= (n+1)a^\dagger|n\rangle \\
Na|n\rangle &= ([N, a] + aN)|n\rangle \\
&= (-a + an)|n\rangle \\
&= (n-1)a|n\rangle
\end{aligned}$$

we can see that $a^\dagger|n\rangle \propto |n+1\rangle$ and $a|n\rangle \propto |n-1\rangle$: $a^\dagger|n\rangle = c_+|n+1\rangle$ and $a|n\rangle = c_-|n-1\rangle$. Using the requirement that the eigenstates be normalized, we have

$$\begin{aligned}
\Big(\langle n|a\Big)\Big(a^\dagger|n\rangle\Big) &= \Big({c_+}^*\langle n+1|\Big)\Big(c_+|n+1\rangle\Big) = |c_+|^2 \\
&= \left\langle n\middle|aa^\dagger\middle|n\right\rangle = \left\langle n\middle|\Big([a, a^\dagger] + a^\dagger a\Big)\middle|n\right\rangle = \langle n|1 + N|n\rangle = n + 1 \\
\Big(\langle n|a^\dagger\Big)\Big(a|n\rangle\Big) &= \Big({c_-}^*\langle n-1|\Big)\Big(c_-|n-1\rangle\Big) = |c_-|^2 \\
&= \left\langle n\middle|a^\dagger a\middle|n\right\rangle = \langle n|N|n\rangle = n
\end{aligned}$$

By convention, we let c_+ and c_- be real and positive, so $c_+ = \sqrt{n+1}$ and $c_- = \sqrt{n}$, and thus

$$a^\dagger|n\rangle = \sqrt{n+1}\;|n+1\rangle \qquad a|n\rangle = \sqrt{n}\;|n-1\rangle \tag{5}$$

Since it is required that the norm of any state (say, $a|n\rangle$) be positive, it must be that

$$n = \langle n|N|n\rangle = \Big(\langle n|a^\dagger\Big)\Big(a|n\rangle\Big) \geq 0$$

So, it is said[3], there must be some reason that the annihilator a cannot bring n below zero. Mathematically, we give a reason, that n must be a non-negative integer, $n \in \mathbb{N} = \{0, 1, 2, 3, ...\}$, so that when a operates on $|0\rangle$ it is not allowed to "generate" a lower state:

$$a|0\rangle = (0)|-1\rangle \equiv \mathbf{0} \equiv |\rangle$$

[3] I would leave out the "it is said" if I knew what the *physical* significance of applying the creation and annihilation operators.

$$N\left|0\right\rangle = (0)\left|0\right\rangle \equiv \mathbf{0} \equiv \left|\right\rangle$$

This "state" $\left|\right\rangle = \mathbf{0}$ is not normalizable[4]

$$\begin{aligned}\left\langle \middle| \right\rangle &= \Big(\left\langle 0\right| a^\dagger\Big)\Big(a\left|0\right\rangle\Big) = \left\langle 0 \middle| a^\dagger a \middle| 0\right\rangle = \left\langle 0 \middle| N \middle| 0\right\rangle = 0\left\langle 0|0\right\rangle = (0)(1) = 0 \\ &= \Big(\left\langle 0\right| N^\dagger\Big)\Big(N\left|0\right\rangle\Big) = |0|^2\left\langle 0|0\right\rangle = (0)(1) = 0\end{aligned}$$

and is not usually mentioned[5] when doing the quantum simple harmonic oscillator problem. We will use in the next section this property of the null state $\left|\right\rangle$:

$$\left\langle x \middle| \right\rangle = 0$$

Anyway, then, the ground state is $\left|0\right\rangle$ with energy $E_0 = \frac{1}{2}\hbar\omega$, and the higher energy states $\left|n\right\rangle$, with energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ where $n \in \mathbb{N}$, may be obtained by applying the creation operator, scaled for normalization:

$$\begin{aligned}\left|1\right\rangle &= \left(\frac{a^\dagger}{\sqrt{1}}\right)\left|0\right\rangle \\ \left|2\right\rangle &= \left(\frac{a^\dagger}{\sqrt{2}}\right)\left|1\right\rangle = \left(\frac{{a^\dagger}^2}{\sqrt{2!}}\right)\left|0\right\rangle \\ \left|3\right\rangle &= \left(\frac{a^\dagger}{\sqrt{3}}\right)\left|2\right\rangle = \left(\frac{{a^\dagger}^3}{\sqrt{3!}}\right)\left|0\right\rangle \\ &\vdots \\ \left|n\right\rangle &= \left(\frac{{a^\dagger}^n}{\sqrt{n!}}\right)\left|0\right\rangle \end{aligned} \tag{6}$$

Using equations 5 and the orthonormality of the set $\{\left|n\right\rangle\}_{\forall n}$, we have

$$\left\langle n' \middle| a^\dagger \middle| n\right\rangle = \sqrt{n+1}\,\delta_{n',n+1} \qquad \left\langle n' \middle| a \middle| n\right\rangle = \sqrt{n}\,\delta_{n',n-1}$$

or, in matrix form, represented in the $\left|n\right\rangle$ basis,

$$a^\dagger \doteq \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{n+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad a \doteq \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{n} & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

[4]There are other non-normalizable states – plane wave states in free space, which are "Dirac-normalized". What if we allowed $\mathbf{0}' = \mathbf{0}/0$ to be similarly quasi-normalized?

[5]I've never seen any books address the "state" that you get when you multiply a state $\left|n\right\rangle$ by zero. In linear algebra, when you multiply a vector by the zero scalar, you get the zero vector, *not* the zero scalar. I made up the notation $(0)\left|n\right\rangle = \mathbf{0}$. This issue may disappear when you just look at the differential equation and solve it analytically. In the Wikipedia article on "creation and annihilation operators", I've seen this state referred to as the null state, empty state, or vacuum state $\left|\right\rangle$, which is seen in quantum field theory. So perhaps this is conventional, but rarely written: $(0)\left|n\right\rangle = \left|\right\rangle$.

Using equations 4, we have

$$\langle n'|X|n\rangle = x_0\left(\sqrt{n+1}\,\delta_{n',n+1} + \sqrt{n}\,\delta_{n',n-1}\right) \qquad \langle n'|P|n\rangle = ip_0\left(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right)$$

Note that for all n,

$$\langle X\rangle_n \equiv \langle n|X|n\rangle = 0 \qquad \langle P\rangle_n \equiv \langle n|P|n\rangle = 0$$

2.5 Energy Eigenfunctions

We can use the annihilator a and creator $a^\dagger$ to find the energy eigenfunctions in the position representation (as well as the momentum representation). (Let $\mathrm{d}_x = \mathrm{d}/\mathrm{d}x$.)

$$\begin{aligned}\langle x|a|0\rangle &= \frac{1}{2}\left\langle x\left|\left(\frac{X}{x_0}+i\frac{P}{p_0}\right)\right|0\right\rangle = \frac{1}{2}\left(\frac{x}{x_0}+i(-i\hbar)\frac{\mathrm{d}_x}{p_0}\right)\langle x|0\rangle = \frac{1}{2}\left(\frac{x}{x_0}+(2x_0p_0)\frac{\mathrm{d}_x}{p_0}\right)\langle x|0\rangle \\ &= \langle x|\rangle = 0\end{aligned}$$

So

$$\left(x + 2{x_0}^2\mathrm{d}_x\right)\langle x|0\rangle = 0$$

and the normalized solution is

$$\langle x|0\rangle = \frac{1}{\pi^{1/4}\sqrt{\sqrt{2}\,x_0}}\exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{2}\,x_0}\right)^2\right] = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}}\exp\left[-\frac{1}{4}\left(\frac{x}{x_0}\right)^2\right]$$

so that the probability density

$$\left|\langle x|0\rangle\right|^2 = \frac{1}{\sqrt{2\pi}\,x_0}\exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

is a Gaussian distribution over space with a standard deviation of x_0.

It is useful to know the form that $a^\dagger$ and a take in the position and momentum representations. Using equations 1, 2, and 3, we get

$$\langle x|\,a^\dagger = \frac{1}{2}\left(\frac{x}{x_0} - 2x_0\,\mathrm{d}_x\right)\langle x| \qquad \langle x|\,a = \frac{1}{2}\left(\frac{x}{x_0} + 2x_0\,\mathrm{d}_x\right)\langle x|$$

$$\langle p|\,a^\dagger = \frac{i}{2}\left(2p_0\,\mathrm{d}_p - \frac{p}{p_0}\right)\langle p| \qquad \langle p|\,a = \frac{i}{2}\left(2p_0\,\mathrm{d}_p + \frac{p}{p_0}\right)\langle p|$$

In the position basis, we may use the creation operator to find expressions for the other eigenfunctions as is shown in equation 6:

$$\begin{aligned}\langle x|n\rangle &= \left\langle x\left|\frac{{a^\dagger}^n}{\sqrt{n!}}\right|0\right\rangle = \frac{1}{\sqrt{n!}}\left[\frac{1}{2}\left(\frac{x}{x_0} - 2x_0\,\mathrm{d}_x\right)\right]^n\langle x|0\rangle \\ &= \left(\frac{1}{(2\pi)^{1/4}\,2^n\sqrt{n!}}\right)\left(\frac{1}{{x_0}^{n+1/2}}\right)\left(x - 2{x_0}^2\mathrm{d}_x\right)^n\exp\left[-\frac{1}{4}\left(\frac{x}{x_0}\right)^2\right]\end{aligned}$$

2.6 Uncertainty (Variance) Products

We also have

$$X^2 = {x_0}^2 \left({a^\dagger}^2 + a^\dagger a + a a^\dagger + a^2 \right) \qquad P^2 = -{p_0}^2 \left({a^\dagger}^2 - a^\dagger a - a a^\dagger + a^2 \right)$$

so that

$$\left\langle X^2 \right\rangle_n = \left\langle n \middle| X^2 \middle| n \right\rangle = {x_0}^2 \left(0 + (\sqrt{n}\sqrt{n}) + \left(\sqrt{n+1}\sqrt{n+1}\right) + 0 \right) = {x_0}^2(2n+1)$$

$$\left\langle P^2 \right\rangle_n = \left\langle n \middle| P^2 \middle| n \right\rangle = -{p_0}^2 \left(0 - (\sqrt{n}\sqrt{n}) - \left(\sqrt{n+1}\sqrt{n+1}\right) + 0 \right) = {p_0}^2(2n+1)$$

Since we have that the variance (or "uncertainty") $\langle (\Delta X)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2 = \langle X^2 \rangle$, and the same for P, we also have

$$\left\langle (\Delta X)^2 \right\rangle_n \left\langle (\Delta P)^2 \right\rangle_n = {x_0}^2 {p_0}^2 (2n+1)^2 = \frac{\hbar^2}{4}(2n+1)^2 = \hbar^2 \left(n + \tfrac{1}{2} \right)^2$$

Note that the ground state (which happens to be a Gaussian curve in position and momentum) yields the absolute minimum uncertainty

$$\left\langle (\Delta X)^2 \right\rangle_0 \left\langle (\Delta P)^2 \right\rangle_0 = \frac{\hbar^2}{4}$$

and the uncertainty products for the excited states are larger.

2.7 Selection Rules for Creation and Annihilation

For an operator that has a term T that contains the $a^\dagger$ operator r times and the a operator s times,

$$\langle n' | T | n \rangle = 0$$

unless $n' = n - s + r$. For example, ${a^\dagger}^2 a^3$ and $a^\dagger a^2 a^\dagger a$, both contain $a^\dagger$ 2 times and a 3 times, and so they connect states that are one quantum apart: $\left\langle 4 \middle| {a^\dagger}^2 a^3 \middle| 5 \right\rangle \neq 0$ and $\left\langle 4 \middle| a^\dagger a^2 a^\dagger a \middle| 5 \right\rangle \neq 0$.

However, if the ordering of $a^\dagger$ and a in T is such that it lowers $|n\rangle$ to the null state $|\rangle$, then

$$\langle n' | T | n \rangle = 0$$

For example, $\left\langle 1 \middle| {a^\dagger}^2 a^3 \middle| 2 \right\rangle = 0$ but $\left\langle 1 \middle| a {a^\dagger}^2 a^2 \middle| 2 \right\rangle \neq 0$.

Example Problem

Oscillator with Cubic Perturbation
(Abers [2] Problem 7.2, from Problem Set 11, problem number 48)

Question: Consider a force that increases quadratically with distance from the origin as a perturbation to the one-dimensional harmonic oscillator; that is, $H = H_0 + H^1$, where

$$H_0 = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 \quad \text{and} \quad H^1 = -\beta X^3$$

Find the shifts of the first two energy levels through the lowest nonvanishing order in perturbation theory.

Answer: The unperturbed energies are $\hbar\omega\left(n+\frac{1}{2}\right)$. The perturbation is

$$\begin{aligned}
H^1 &= -\beta X^3 = -\beta {x_0}^3 (a^\dagger + a)^3 \\
&= -\beta {x_0}^3 ({a^\dagger}^2 + aa^\dagger + a^\dagger a + a^2)(a^\dagger + a) \\
&= -\beta {x_0}^3 ({a^\dagger}^3 + a{a^\dagger}^2 + a^\dagger a a^\dagger + a^2 a^\dagger + {a^\dagger}^2 a + a^\dagger a^2 + a a^\dagger a + a^3)
\end{aligned}$$

so the perturbation connects states whose levels differ by ± 1 or ± 3; it has no diagonal elements, so there is no first-order energy shift ($\Delta_n^{(1)} = \langle n|H^1|n\rangle = 0$). We'll have to use second order perturbation[6] calculations:

$$\Delta_n^{(2)} = -\sum_{m\neq n} \frac{\left|\langle m|H^1|n\rangle\right|^2}{E_m^0 - E_n^0}$$

The matrix elements needed are

$$\begin{aligned}
\left\langle 1\middle|H^1\middle|0\right\rangle &= -\beta {x_0}^3 \left\langle 1\middle|\left(a{a^\dagger}^2 + a^\dagger a a^\dagger\right)\middle|0\right\rangle &=& -(\sqrt{2^2}+\sqrt{1})\,\beta {x_0}^3 = &-3\,\beta {x_0}^3 \\
\left\langle 3\middle|H^1\middle|0\right\rangle &= -\beta {x_0}^3 \left\langle 3\middle|{a^\dagger}^3\middle|0\right\rangle &=& -\sqrt{2\cdot 3}\,\beta {x_0}^3 = &-\sqrt{6}\,\beta {x_0}^3 \\
\left\langle 0\middle|H^1\middle|1\right\rangle &= -\beta {x_0}^3 \left\langle 0\middle|\left(a^2 a^\dagger + a a^\dagger a\right)\middle|1\right\rangle &=& -(\sqrt{2^2}+\sqrt{1})\,\beta {x_0}^3 = &-3\,\beta {x_0}^3 \\
\left\langle 2\middle|H^1\middle|1\right\rangle &= -\beta {x_0}^3 \left\langle 2\middle|\left(a{a^\dagger}^2 + a^\dagger a a^\dagger + {a^\dagger}^2 a\right)\middle|1\right\rangle &=& -(\sqrt{2\cdot 3^2}+\sqrt{2^3}+\sqrt{2})\,\beta {x_0}^3 = &-6\sqrt{2}\,\beta {x_0}^3 \\
\left\langle 4\middle|H^1\middle|1\right\rangle &= -\beta {x_0}^3 \left\langle 4\middle|{a^\dagger}^3\middle|1\right\rangle &=& -\sqrt{2\cdot 3\cdot 4}\,\beta {x_0}^3 = &-2\sqrt{6}\,\beta {x_0}^3
\end{aligned}$$

So the second-order shifts of the ground state energy and the first excited state energy are

$$\begin{aligned}
\Delta_0^{(2)} &= -\sum_{m\neq 0} \frac{\left|\langle m|H^1|0\rangle\right|^2}{E_m^0 - E_0^0} \\
&= -\left(\beta {x_0}^3\right)^2 \left[\frac{(3)^2}{\hbar\omega} + \frac{(\sqrt{6})^2}{3\hbar\omega}\right] = -11\frac{\left(\beta {x_0}^3\right)^2}{\hbar\omega} \\
&= -11\frac{\beta^2}{\hbar\omega}\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^6 = -\frac{11}{8}\frac{\beta^2\hbar^2}{m^3\omega^4} \\
\Delta_1^{(2)} &= -\sum_{m\neq 1} \frac{\left|\langle m|H^1|1\rangle\right|^2}{E_m^0 - E_1^0} \\
&= -\left(\beta {x_0}^3\right)^2 \left[\frac{(3)^2}{-\hbar\omega} + \frac{(6\sqrt{2})^2}{\hbar\omega} + \frac{(2\sqrt{6})^2}{3\hbar\omega}\right] = -\left(\beta {x_0}^3\right)^2 \left[-\frac{9}{\hbar\omega} + \frac{72}{\hbar\omega} + \frac{8}{\hbar\omega}\right] = -71\frac{\left(\beta {x_0}^3\right)^2}{\hbar\omega} \\
&= -71\frac{\beta^2}{\hbar\omega}\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^6 = -\frac{71}{8}\frac{\beta^2\hbar^2}{m^3\omega^4}
\end{aligned}$$

[6] Perturbation theory (and other topics) are explained well in Griffiths' quantum mechanics text [3].

3 Coupled Harmonic Oscillators

4 Landau Levels

4.1 2-D Harmonic Oscillator due to a Magnetic Field

See Abers [2] Section 6.1.3.

$$[L_i, a_j^\dagger] = i \sum_k \epsilon_{ijk} a_k^\dagger$$

$$a_\pm^\dagger \equiv \frac{1}{\sqrt{2}} \left(a_x^\dagger \pm i a_y^\dagger \right)$$

$$a_\pm \equiv \frac{1}{\sqrt{2}} \left(a_x \mp i a_y \right)$$

$$[a_-, a_+^\dagger] = 0$$

$$[L_z, a_\pm^\dagger] = \pm a_\pm^\dagger$$

References

[1] J. J. Sakurai: *Modern Quantum Mechanics*, Addison-Wesley (1994)

[2] Ernest S. Abers: *Quantum Mechanics*, Pearson, Prentice Hall (2004)

[3] David Griffiths: *Introduction to Quantum Mechanics, Second Edition*, Pearson Education, Inc. (2005)