Bonus Lecture: Concepts of Quantum Field Theory (QFT)

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Further Investigation of an Intuitive Approach to QFT

- Is my intuitive picture of QFT true?

$$\Psi = \Psi(\mathbf{x},\phi) \qquad \Psi_0(\mathbf{x},\phi) = \frac{1}{(2\pi)^{1/4}\sqrt{\phi_0}}\, e^{-\frac{1}{4}(\phi/\phi_0)^2}$$

$$\Psi[\varphi] = \prod_{\mathbf{x}} \Psi\big(\mathbf{x},\varphi(\mathbf{x})\big) = \prod_{\mathbf{x}} e^{\ln\Psi\left(\mathbf{x},\varphi(\mathbf{x})\right)} = e^{\sum_{\mathbf{x}} \ln\Psi\left(\mathbf{x},\varphi(\mathbf{x})\right)} = e^{\int \mathrm{d}^3x\, \ln\Psi\left(\mathbf{x},\varphi(\mathbf{x})\right)}$$

$$\Psi_0[\varphi] = e^{-\frac{1}{2}\ln\left(\sqrt{2\pi}\,\phi_0\right)\int \mathrm{d}^3x}\; e^{-\frac{1}{4}\int \mathrm{d}^3x\, \varphi^2(\mathbf{x})/\phi_0{}^2}$$

  - What concepts will make our equation be the same as that in Hatfield, using Eqns. (10.15) and (10.17):

$$\Psi_0[\varphi] = \eta\, e^{-\int \mathrm{d}^3x\,\mathrm{d}^3y\;\; \varphi(\mathbf{x})\, g(\mathbf{x},\mathbf{y})\, \varphi(\mathbf{y})} \qquad ?$$

  or Eqn. (10.26), the equation above (10.23), and (10.28):

$$\Psi_0[\tilde{\varphi}] = \eta\, e^{-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\;\; \omega_k\, \tilde{\varphi}(\mathbf{k})\, \tilde{\varphi}(-\mathbf{k})} \qquad ?$$

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2}$$

$$\eta = \prod_{\mathbf{k}} \left(\frac{\omega_k}{\pi}\right)^{1/4} = e^{\int \mathrm{d}^3k\, \ln\left(\sqrt{\mathbf{k}^2+m^2}/\pi\right)^{1/4}} = e^{\frac{1}{4}\int \mathrm{d}^3k\, \ln\left(\sqrt{\mathbf{k}^2+m^2}/\pi\right)}$$

  - What is $g(\mathbf{x},\mathbf{y})$? Judging by its Fourier transform, $\tilde{g}(\mathbf{k}) = \sqrt{\mathbf{k}^2+m^2}/2$, it does not exist! What could this mean?
  - My intuitive idea uses ground wave-functions $\Psi_0(\mathbf{x},\phi) = \Psi_0(\phi)$ of independent oscillators. Since they are all continuously coupled, this may be the source of the function $g(\mathbf{x},\mathbf{y})$ that mixes the fields up in the wave-functional.

- Are general states Ψ proportional to the ground state? See Hatfield Eqns. (10.6) and (10.30) and Lecture 6. Given that $|\Psi\rangle$ is a QFT eigenstate for a field of configuration $\Psi(x)$, isn't it true that

$$\begin{aligned} \Psi[\varphi] &= \langle\varphi|\Psi\rangle = \langle\varphi|\left(\int \mathrm{d}x\;\Psi(x)\;\hat{\varphi}^\dagger(x)\;|0\rangle\right) = \int \mathrm{d}x\;\Psi(x)\;\left\langle\varphi\middle|\hat{\varphi}^\dagger(x)\middle|0\right\rangle \\ &= \int \mathrm{d}x\;\Psi(x)\;\varphi^*(x)\;\langle\varphi|0\rangle = \left[\int \mathrm{d}x\;\Psi(x)\,\varphi^*(x)\right]\Psi_0[\varphi] \qquad ? \end{aligned}$$

  (Are there typos in Eqn. (10.30)? Shouldn't that be $e^{i\vec{k}_1\cdot\vec{y}}$ and $\phi^*(\vec{y})$?)

  This seems to be quite different from the above intuitive proposal for general states.

- Given $F(\phi,\mathbf{x})$ and $F(\varphi(\mathbf{x}),\mathbf{x}) = F_{\mathbf{x}}(\varphi(\mathbf{x}))$ with $F[\varphi] = \int \mathrm{d}^3x\; F(\varphi(\mathbf{x}),\mathbf{x})$, how do

$$\frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})} \quad \text{and} \quad \frac{\partial F}{\partial\phi}$$

  relate?

- What is the solution to the functional differential equation from lecture 9?

$$\sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}}\frac{\delta}{\delta\tilde{\phi}(\mathbf{p})}\right)\Phi_0[\tilde{\phi}] = 0$$

Continuous Sums and Products

Let's examine the flaws in the following mathematical relations that are seen in Hatfield's book [1]:

$$\prod_x f(x) = \prod_x e^{\ln f(x)} = e^{\sum_x \ln f(x)} = e^{\int \mathrm{d}x \, \ln f(x)}. \tag{1}$$

The last two equalities are seen in Hatfield Equation (9.17). The basic assumption here is that a "continuous sum" is the same as an integral:

$$\sum_x g(x) = \int \mathrm{d}x \; g(x),$$

but this is not true, strictly or naïvely speaking.

A "continuous sum" is not really an integral. Let's take for example a function $f(x)$ on the real line that is equal to 0 everywhere except for a small region where it jumps continuously up to 1.1 and falls back down continuously to 0. If we examine the continuous sum of this function

$$\sum_x f(x),$$

which is the sum of the values of $f(x)$ for each and every x on the real line, we'll find that it diverges, while the integral of this function is finite. We can find two points $x = a$ and $x = b$ where the function is equal to 1, where $a < b$. For all points x in the region (a, b), $f(x)$ is greater than 1. We can find countably infinitely many points in this region; we can pick the midpoint m of (a, b), then the two midpoints of (a, m) and (m, b), and then the four next midpoints, and so on. Thus the continuous sum includes a sum of countably infinitely many numbers that are greater than 1. This sum diverges, and so therefore the continuous sum diverges. This argument, in fact, works for any function that continuously deviates from 0, no matter how little, unless it deviates equally above and below 0. Also, the same kind of argument works for a continuous product:

$$\prod_x f(x).$$

The continuous product of a function that continuously deviates from 1 is either undefined or 0 (unless it deviates compensatingly above and below 1).

So now we know we've made some error in writing the string of relations (1) above. It seems that the ideas of a "continuous sum" and a "continuous product" are on shaky ground, since they are so delicate (one small deviation and they explode or implode). So we should work backwards from the last expression in (1), which seems to be what we want, and find what we really wanted to start with:

$$e^{\int \mathrm{d}x \, \ln f(x)} \to e^{\sum_i \Delta x_i \, \ln f(x_i)} = \prod_i e^{\Delta x_i \, \ln f(x_i)} = \prod_i e^{\ln f(x_i)^{\Delta x_i}} = \prod_i f(x_i)^{\Delta x_i}.$$

Maybe the "continuous product" (and the "continuous sum" along with it) needs to be corrected to be of use. Instead of using the naïve definition of the continuous product, we will use

$$\prod_x f(x) \quad \equiv \quad \lim_{\epsilon \to 0} \prod_i f(x_i)^{\Delta x_i} \quad = \quad e^{\int \mathrm{d}x \, \ln f(x)},$$

where ϵ is the mesh or norm of the partition (x_i), meaning of all the intervals $[x_i, x_{i+1}]$, ϵ is the length of the largest of those intervals. With this definition, one would then naturally associate the continuous sum with the integral, for consistency and convenience:

$$\sum_x f(x) \quad \equiv \quad \lim_{\epsilon \to 0} \sum_i \Delta x_i \, f(x_i) \quad = \quad \int \mathrm{d}x \; f(x).$$

Note that both x and f should properly take unitless values. The only remaining matter is to investigate how well-defined or robust these concepts are.

My Naming Conventions

- φ: field (as a function)
- ϕ: field coordinate
- $\hat{\varphi}$: field operator
- $\hat{\phi}$: field coordinate operator (is this used?)
- $\Psi(\mathbf{x}, \phi)$: state-function (field-wave-function, meta-wave-function)
- $\Psi[\varphi]$: wave-functional (or state-functional)
- $\Psi_{\mathbf{x}}(\phi)$: "point-wave-function?" ($= \Psi(\mathbf{x}, \phi)$, considering $\mathbf{x}$ constant and ϕ variable)
  This is the wave-function of an "oscillator" at a given point $\mathbf{x}$.
- $\Psi_{\phi}(\mathbf{x})$: "displacement-wave-function?" ($= \Psi(\mathbf{x}, \phi)$, considering ϕ constant and $\mathbf{x}$ variable)
- $\Psi_{\varphi}(\mathbf{x})$: etc.

Functional Normalization

From Hatfield (10.25) and the assumption $\Psi[\varphi] = \prod_{\mathbf{x}} \Psi_{\mathbf{x}}(\varphi(\mathbf{x}))$ we infer

$$\begin{aligned}
1 &= \int \mathcal{D}\varphi \, \big|\Psi[\varphi]\big|^2 \\
&= \int \prod_{\mathbf{x}} \delta\varphi(\mathbf{x}) \, \big|\Psi[\varphi]\big|^2 \\
&= \int \prod_{\mathbf{x}} \delta\varphi(\mathbf{x}) \left[\prod_{\mathbf{y}} \Psi^*_{\mathbf{y}}\big(\varphi(\mathbf{y})\big)\right] \left[\prod_{\mathbf{z}} \Psi_{\mathbf{z}}\big(\varphi(\mathbf{z})\big)\right] \\
&= \int \prod_{\mathbf{x}} \delta\varphi(\mathbf{x}) \; \Psi^*_{\mathbf{x}}\big(\varphi(\mathbf{x})\big) \; \Psi_{\mathbf{x}}\big(\varphi(\mathbf{x})\big) \\
&= \prod_{\mathbf{x}} \left(\int \delta\varphi(\mathbf{x}) \; \Psi^*_{\mathbf{x}}\big(\varphi(\mathbf{x})\big) \; \Psi_{\mathbf{x}}\big(\varphi(\mathbf{x})\big) \right) \\
&= \prod_{\mathbf{x}} \left(\int \mathrm{d}\phi \; \Psi^*_{\mathbf{x}}(\phi) \, \Psi_{\mathbf{x}}(\phi) \right) \\
&= \prod_{\mathbf{x}} \left(\int \mathrm{d}\phi \, \big|\Psi_{\mathbf{x}}(\phi)\big|^2 \right) \\
\Rightarrow \qquad 1 &= \int \mathrm{d}\phi \, \big|\Psi_{\mathbf{x}}(\phi)\big|^2 \qquad \forall \mathbf{x}
\end{aligned}$$

The wave-function $\Psi_{\mathbf{x}}$ for each "oscillator" in space is normalized to 1.

QFT Probabilities

Following the pattern above, the probability P that the field is contained within the extrema fields φ_1 and φ_2, where $\varphi_1(\mathbf{x}) < \varphi_2(\mathbf{x})$ for all $\mathbf{x}$, could be

$$P(\varphi_1, \varphi_2) \;=\; \int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi \, \big|\Psi[\varphi]\big|^2 \;=\; \prod_{\mathbf{x}} \left(\int_{\varphi_1(\mathbf{x})}^{\varphi_2(\mathbf{x})} \mathrm{d}\phi \, \big|\Psi_{\mathbf{x}}(\phi)\big|^2 \right) \;=\; \exp\left[\int \mathrm{d}^3x \; \ln \left(\int_{\varphi_1(\mathbf{x})}^{\varphi_2(\mathbf{x})} \mathrm{d}\phi \, \big|\Psi_{\mathbf{x}}(\phi)\big|^2 \right) \right].$$

And, as seen above, the probability that the field takes some configuration is one:

$$P(-\infty, \infty) \;=\; \prod_{\mathbf{x}} \left(\int_{-\infty}^{\infty} \mathrm{d}\phi \, \big|\Psi_{\mathbf{x}}(\phi)\big|^2 \right) \;=\; 1.$$

Field Functional-Derivative Versus Field-Coordinate Partial-Derivative

Let $F(\mathbf{x},\phi) = F_{\mathbf{x}}(\phi)$ and $F\big(\mathbf{x},\varphi(\mathbf{x})\big) = F_{\mathbf{x}}\big(\varphi(\mathbf{x})\big)$ with $F[\varphi] = \int \mathrm{d}^3x \; F\big(\mathbf{x},\varphi(\mathbf{x})\big)$. How do

$$\frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})} \qquad \text{and} \qquad \frac{\partial F}{\partial \phi}$$

relate? They relate in this way:

$$\begin{aligned}
\frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})} &= \frac{\delta}{\delta\varphi(\mathbf{x})} \int \mathrm{d}^3y \; F\big(\mathbf{y},\varphi(\mathbf{y})\big) \\
&= \int \mathrm{d}^3y \; \frac{\delta}{\delta\varphi(\mathbf{x})} \, F\big(\mathbf{y},\varphi(\mathbf{y})\big) \\
&= \int \mathrm{d}^3y \; \frac{\partial F(\mathbf{y},\phi)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{y})} \delta(\mathbf{y}-\mathbf{x}) \\
&= \frac{\partial F(\mathbf{x},\phi)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{x})}
\end{aligned}$$

$$\boxed{\frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})} = \frac{\partial F(\mathbf{x},\phi)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{x})}}$$

Now, let $\Psi(\mathbf{x},\phi) = \Psi_{\mathbf{x}}(\phi)$ and $\Psi\big(\mathbf{x},\varphi(\mathbf{x})\big) = \Psi_{\mathbf{x}}\big(\varphi(\mathbf{x})\big)$ with

$$\Psi[\varphi] = e^{\int \mathrm{d}^3x \; \ln \Psi\left(\mathbf{x},\varphi(\mathbf{x})\right)}.$$

How do

$$\frac{\delta \Psi[\varphi]}{\delta\varphi(\mathbf{x})} \qquad \text{and} \qquad \frac{\partial \Psi}{\partial \phi}$$

relate? They relate in this way:

$$\begin{aligned}
\frac{\delta \Psi[\varphi]}{\delta\varphi(\mathbf{x})} &= \frac{\delta}{\delta\varphi(\mathbf{x})} \, e^{\int \mathrm{d}^3y \; \ln \Psi\left(\mathbf{y},\varphi(\mathbf{y})\right)} \\
&= e^{\int \mathrm{d}^3y \; \Psi\left(\mathbf{y},\varphi(\mathbf{y})\right)} \, \frac{\delta}{\delta\varphi(\mathbf{x})} \int \mathrm{d}^3y \; \ln \Psi\big(\mathbf{y},\varphi(\mathbf{y})\big) \\
&= \Psi[\varphi] \int \mathrm{d}^3y \; \frac{\partial\left(\ln \Psi(\mathbf{y},\phi)\right)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{y})} \delta(\mathbf{y}-\mathbf{x}) \\
&= \Psi[\varphi] \; \frac{\partial\left(\ln \Psi(\mathbf{x},\phi)\right)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{x})}
\end{aligned}$$

So

$$\boxed{\frac{\delta \Psi[\varphi]}{\delta\varphi(\mathbf{x})} = \Psi[\varphi] \; \frac{\partial\left(\ln \Psi_{\mathbf{x}}(\phi)\right)}{\partial\phi}\bigg|_{\phi=\varphi(\mathbf{x})}}$$

I think this will be relevant to the intuitive picture of the QFT field.

Attempt 1: Calculating the Ground State-Function, Using Intuitive Assumption

Assuming

$$\begin{aligned}
\Psi[\varphi] &= e^{\int \mathrm{d}^3x \; \ln \Psi_{\mathbf{x}}\left(\varphi(\mathbf{x})\right)} \\
\Psi^0_{\mathbf{x}}(\phi) &= \frac{1}{(2\pi)^{1/4}\sqrt{\phi_0}}\, e^{-\frac{1}{4}(\phi/\phi_0)^2} \quad \equiv \quad \Psi_0(\phi)
\end{aligned}$$

then

$$\begin{aligned}
\frac{\delta\Psi_0[\varphi]}{\delta\varphi(\mathbf{x})} &= \Psi_0[\varphi]\; \frac{\partial\left(\ln\Psi_0(\phi)\right)}{\partial\phi}\Bigg|_{\phi=\varphi(\mathbf{x})} \\
&= \Psi_0[\varphi]\left[\frac{\partial}{\partial\phi}\left(\ln\left(\frac{1}{(2\pi)^{1/4}\sqrt{\phi_0}}\right) - \frac{\phi^2}{4{\phi_0}^2}\right)\right]_{\phi=\varphi(\mathbf{x})} \\
&= -\Psi_0[\varphi]\,\frac{\varphi(\mathbf{x})}{2{\phi_0}^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\delta^2\Psi_0[\varphi]}{\delta\varphi(\mathbf{x})^2} &= \frac{\delta}{\delta\varphi(\mathbf{x})}\left\{-\Psi_0[\varphi]\,\frac{\varphi(\mathbf{x})}{2{\phi_0}^2}\right\} \\
&= \Psi_0[\varphi]\left(-\frac{\varphi(\mathbf{x})}{2{\phi_0}^2}\right)^2 - \Psi_0[\varphi]\;\frac{\delta}{\delta\varphi(\mathbf{x})}\left(\frac{\varphi(\mathbf{x})}{2{\phi_0}^2}\right) \\
&= \Psi_0[\varphi]\left[\frac{\varphi^2(\mathbf{x})}{4{\phi_0}^4} - \frac{\delta(\mathbf{0})}{2{\phi_0}^2}\right]
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{2}\int \mathrm{d}^3x\left\{-\frac{\delta^2\Psi_0[\varphi]}{\delta\varphi(\mathbf{x})^2} + \left(|\boldsymbol{\nabla}\varphi|^2 + m^2\varphi^2\right)\Psi_0[\varphi]\right\} &= E_0\,\Psi_0[\varphi] \\
\frac{1}{2}\int \mathrm{d}^3x\left\{-\cancel{\Psi_0[\varphi]}\left[\frac{\varphi^2}{4{\phi_0}^4} - \frac{\delta(\mathbf{0})}{2{\phi_0}^2}\right] + \left(|\boldsymbol{\nabla}\varphi|^2 + m^2\varphi^2\right)\cancel{\Psi_0[\varphi]}\right\} &= E_0\,\cancel{\Psi_0[\varphi]} \\
\frac{1}{2}\int \mathrm{d}^3x\left\{\frac{\delta(\mathbf{0})}{2{\phi_0}^2} - \frac{\varphi^2}{4{\phi_0}^4} + \left(|\boldsymbol{\nabla}\varphi|^2 + m^2\varphi^2\right)\right\} &= E_0 \\
-\frac{\delta(\mathbf{0})}{4{\phi_0}^2}\int \mathrm{d}^3x + \frac{1}{8{\phi_0}^4}\int \mathrm{d}^3x\;\varphi^2(\mathbf{x}) &= E_0 - \frac{1}{2}\int \mathrm{d}^3x\;\varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2 + m^2\right\}\varphi(\mathbf{x}) \\
-\frac{\delta^2(\mathbf{0})}{4{\phi_0}^2} + \frac{1}{8{\phi_0}^4}\int \mathrm{d}^3x\;\varphi^2(\mathbf{x}) &= E_0 - \frac{1}{2}\int \mathrm{d}^3x\;\varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2 + m^2\right\}\varphi(\mathbf{x})
\end{aligned}$$

$$\boxed{\begin{aligned}
E_0 &= -\frac{\delta^2(\mathbf{0})}{4{\phi_0}^2} \\
\frac{1}{4{\phi_0}^4}\int \mathrm{d}^3x\;\varphi^2(\mathbf{x}) &= \int \mathrm{d}^3x\;\varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2 + m^2\right\}\varphi(\mathbf{x})
\end{aligned}}$$

This could be right, but it probably isn't. (Compare this with Hatfield (10.18), (10.19), and the analysis suronding those equations.)

Attempt 2: Calculating the Ground State-Function, Using Intuitive Assumption

Instead of using

$$\frac{1}{2}\int \mathrm{d}^3x \left(-\frac{\delta^2\Psi_0[\varphi]}{\delta\varphi(\mathbf{x})^2}\right),$$

let's use

$$\frac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2 \left(-\frac{\delta^2\Psi_0[\varphi]}{\delta\varphi(\mathbf{x}_1)\,\delta\varphi(\mathbf{x}_2)}\right),$$

so we have less chance of finding divergences. I don't think this is right, though, because the assignment

$$\pi(\mathbf{x}) \to \frac{\delta}{\delta\varphi(\mathbf{x})} \qquad \text{and} \qquad \pi^2(\mathbf{x}) \to \frac{\delta^2}{\delta\varphi(\mathbf{x})^2}$$

seems to be right, and divergences seem to be acceptable in QFT, in the right circumstances (such as energy of a field over all space).

Assuming

$$\begin{aligned}
\Psi[\varphi] &= e^{\int \mathrm{d}^3x\ \ln\Psi_{\mathbf{x}}(\varphi(\mathbf{x}))} \\
\Psi^0_{\mathbf{x}}(\phi) &= \frac{1}{(2\pi)^{1/4}\sqrt{\phi_0}}\,e^{-\frac{1}{4}(\phi/\phi_0)^2} \quad \equiv \quad \Psi_0(\phi)
\end{aligned}$$

then

$$\frac{\delta\Psi_0[\varphi]}{\delta\varphi(\mathbf{x}_2)} = -\Psi_0[\varphi]\,\frac{\varphi(\mathbf{x}_2)}{2{\phi_0}^2}$$

and

$$\begin{aligned}
\frac{\delta^2\Psi_0[\varphi]}{\delta\varphi(\mathbf{x}_1)\,\delta\varphi(\mathbf{x}_2)} &= \frac{\delta}{\delta\varphi(\mathbf{x}_1)}\left(-\Psi_0[\varphi]\,\frac{\varphi(\mathbf{x}_2)}{2{\phi_0}^2}\right) \\
&= \left(-\Psi_0[\varphi]\,\frac{\varphi(\mathbf{x}_1)}{2{\phi_0}^2}\right)\left(-\frac{\varphi(\mathbf{x}_2)}{2{\phi_0}^2}\right) - \Psi_0[\varphi]\,\frac{\delta}{\delta\varphi(\mathbf{x}_1)}\left(\frac{\varphi(\mathbf{x}_2)}{2{\phi_0}^2}\right) \\
&= \Psi_0[\varphi]\left[\frac{\varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2)}{4{\phi_0}^4} - \frac{\delta(\mathbf{x}_2-\mathbf{x}_1)}{2{\phi_0}^2}\right]
\end{aligned}$$

so

$$\begin{aligned}
-\frac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\left[\frac{\varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2)}{4{\phi_0}^4} - \frac{\delta(\mathbf{x}_2-\mathbf{x}_1)}{2{\phi_0}^2}\right] &= E_0 - \frac{1}{2}\int \mathrm{d}^3x\ \varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2+m^2\right\}\varphi(\mathbf{x}) \\
\frac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\frac{\delta(\mathbf{x}_2-\mathbf{x}_1)}{2{\phi_0}^2} - \frac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\frac{\varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2)}{4{\phi_0}^4} &= E_0 - \frac{1}{2}\int \mathrm{d}^3x\ \varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2+m^2\right\}\varphi(\mathbf{x}) \\
\frac{1}{4{\phi_0}^2}\int \mathrm{d}^3x_1 - \frac{1}{2}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\frac{\varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2)}{4{\phi_0}^4} &= E_0 - \frac{1}{2}\int \mathrm{d}^3x\ \varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2+m^2\right\}\varphi(\mathbf{x})
\end{aligned}$$

$$\boxed{\begin{aligned}
E_0 &= \frac{1}{4{\phi_0}^2}\int \mathrm{d}^3x_1 = \frac{\delta(\mathbf{0})}{4{\phi_0}^2} \\
\frac{1}{4{\phi_0}^4}\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\ \varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2) &= \int \mathrm{d}^3x\ \varphi(\mathbf{x})\left\{-\boldsymbol{\nabla}^2+m^2\right\}\varphi(\mathbf{x})
\end{aligned}}$$

The field interacts over all distances? This does not look right.

Attempt 3: Calculating the Ground State-Function, Using Field Annihilator

What is the solution to the functional differential equation from lecture 9?

$$\sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\tilde{\varphi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}}\,\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\right)\Psi_0[\tilde{\varphi}] = 0$$

$$\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,\Psi_0[\tilde{\varphi}] = -e_{\mathbf{p}}\,\tilde{\varphi}(\mathbf{p})\,\Psi_0[\tilde{\varphi}]$$

We will use an exponential form

$$\Psi_0[\tilde{\varphi}] = \eta\, e^{-G[\tilde{\varphi}]}.$$

So

$$\begin{aligned}
\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,\Psi_0[\tilde{\varphi}] &= \eta\,\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,e^{-G[\tilde{\varphi}]} \\
&= -\eta\, e^{-G[\tilde{\varphi}]}\,\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,G[\tilde{\varphi}] \\
&= -\left(\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,G[\tilde{\varphi}]\right)\Psi_0[\tilde{\varphi}] \\
&= -e_{\mathbf{p}}\,\tilde{\varphi}(\mathbf{p})\,\Psi_0[\tilde{\varphi}].
\end{aligned}$$

What we need is

$$G[\tilde{\varphi}] = \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\;e_{\mathbf{p}}\,\tilde{\varphi}^2(\mathbf{p})$$

because

$$\begin{aligned}
\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,G[\tilde{\varphi}] &= \frac{1}{2}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\;e_{\mathbf{p}'}\,\frac{\delta}{\delta\tilde{\varphi}(\mathbf{p})}\,\tilde{\varphi}^2(\mathbf{p}') \\
&= \frac{1}{2}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\;e_{\mathbf{p}'}\,2\,\tilde{\varphi}(\mathbf{p}')\,\delta(\mathbf{p}'-\mathbf{p}) \\
&= e_{\mathbf{p}}\,\tilde{\varphi}(\mathbf{p})
\end{aligned}$$

So

$$\boxed{\Psi_0[\tilde{\varphi}] = \eta\,\exp\left(-\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\;e_{\mathbf{p}}\,\tilde{\varphi}^2(\mathbf{p})\right).}$$

This does not quite match what is in Hatfield (10.26)

$$\Psi_0[\tilde{\varphi}] = \eta\,\exp\left(-\frac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\;\omega_k\,\tilde{\varphi}(\mathbf{k})\,\tilde{\varphi}(-\mathbf{k})\right).$$

What must be corrected here?

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

- Chapter 10: Free Fields in the Schrödinger Representation