Week 10 Lecture: Concepts of Quantum Field Theory (QFT)

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Functional Calculus

This Week's Questions/Goals

- Is my intuitive picture of QFT true?

$$\Psi = \Psi(x, \phi)$$

$$\Psi[\varphi] = \prod_x \Psi\big(x, \varphi(x)\big) = \prod_x e^{\ln \Psi\left(x,\varphi(x)\right)} = e^{\int \mathrm{d}x \; \ln \Psi\left(x,\varphi(x)\right)}$$

What concepts will make our equation be the same as that in Hatfield, using Eqns. 10.15 and 10.17:

$$\Psi_0[\varphi] = \eta\, e^{-\int \mathrm{d}^3x\, \mathrm{d}^3y \;\; \varphi(\mathbf{x})\, g(\mathbf{x},\mathbf{y})\, \varphi(\mathbf{y})} \qquad ?$$

My idea may have to be modified to include envelope fields. For a 1-D oscillator, there is a probability density that the oscillator will be at a certain displacement, and there is a probability that it will be in a region between two displacements. The envelope fields are like these two displacements.

- Given $F(\phi, \mathbf{x})$ and $F(\varphi(\mathbf{x}), \mathbf{x}) = F_{\mathbf{x}}(\varphi(\mathbf{x}))$ with $F[\varphi] = \int \mathrm{d}^3x \; F(\varphi(\mathbf{x}), \mathbf{x})$, how do

$$\frac{\delta F[\varphi]}{\delta \varphi(\mathbf{x})} \qquad \text{and} \qquad \frac{\partial F}{\partial \phi}$$

relate?

- How does functional calculus work?
- How do these relate?

$$\frac{\delta_\lambda F[a]}{\delta_\lambda a} \qquad\quad \lambda \cdot \tilde{\nabla} F[a] \quad \left(\text{aka } \tfrac{\delta F[a]}{\delta \lambda} \;\, ?\right) \qquad\quad \frac{\delta_\lambda F[a]}{\delta_\lambda a(x)} \equiv \lim_{\epsilon \to 0} \frac{F[a + \epsilon \lambda \delta_x] - F[a]}{\epsilon}$$

- How does one define a second order functional derivative?
- What is the solution to the functional differential equation from last week?

$$\sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \right) \Phi_0[\tilde{\phi}] = 0$$

Functional Calculus

Functionals (of functions) are abstractly very similar to functions of finite-dimensional vectors. Likewise, functional derivatives and integrals are very similar to vector-component partial derivatives and volume integrals.

Analogy with Functions of Vectors

I'd like to emphasize the vector nature of a function a to make the analogy as clear as possible.

Function(al)	Vector	Unit vector	Component	Component projection	Vector composition
$f(\mathbf{x})$	$\mathbf{x}$	$\hat{\mathbf{e}}_i$	x_i	$x_i = \mathbf{x}\cdot\hat{\mathbf{e}}_i = \sum_j x_j\,\hat{e}_{ij} = \sum_j x_j\,\delta_{ij}$	$\mathbf{x} = \sum_i x_i\,\hat{\mathbf{e}}_i$
$F[a]$	a	δ_x	$a(x)$	$a(x) = a\cdot\delta_x = \int \mathrm{d}\xi\ a(\xi)\,\delta(\xi - x)$	$a = \int \mathrm{d}x\ a(x)\,\delta_x$

Orthonormality	Consistency of projection and composition
$\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j = \delta_{ij}$	$x_i = \mathbf{x}\cdot\hat{\mathbf{e}}_i = (\sum_j x_j\,\hat{\mathbf{e}}_i)\cdot\hat{\mathbf{e}}_i = \sum_j x_j\,\delta_{ij}$
$\delta_x\cdot\delta_y = \delta(x-y)$	$a(x) = a\cdot\delta_x = (\int \mathrm{d}y\ a(y)\,\delta_y)\cdot\delta_x = \int \mathrm{d}y\ a(y)\,\delta(x-y)$

Functional Differentiation and Differentials

A component partial derivative is a transformation that takes a function on V and returns a function on V. (Since it returns an object of the same type that it is given, it is called an operator.)

A functional derivative is a transformation that takes a functional on $\mathcal{A}$ and returns a function (or distribution) on K. Because a functional derivative is not an operator, one cannot, strictly speaking, apply it twice. One has to be careful about creating a second order functional derivative. (True?)

$$
\begin{aligned}
\frac{\partial f(\mathbf{x})}{\partial x_i} &\equiv \lim_{\epsilon\to 0}\frac{f(\mathbf{x}+\epsilon\hat{\mathbf{e}}_i) - f(\mathbf{x})}{\epsilon} \\
\frac{\delta F[a]}{\delta a(x)} &\equiv \lim_{\epsilon\to 0}\frac{F[a+\epsilon\delta_x] - F[a]}{\epsilon} \\
\mathrm{d}f(\mathbf{x}) &= \sum_{i=1}^{d}\left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right)\mathrm{d}x_i \quad = \quad \left\{\boldsymbol{\nabla} f(\mathbf{x})\right\}\cdot\mathrm{d}\mathbf{x} \\
&= \left(\sum_{i=1}^{d}\left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right)\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{d}\mathrm{d}x_j\,\hat{\mathbf{e}}_j\right) \\
\delta F[a] &= \int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\ \delta a(x) \quad = \quad \left\{\tilde{\nabla} F[a]\right\}\cdot\delta a \\
&= \left(\int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\ \delta_x\right)\cdot\left(\int \mathrm{d}y\ \delta a(y)\,\delta_y\right)
\end{aligned}
$$

Componentwise derivative	Vector derivative	(bad notation)	Differential (1-form field)	Differential vector
$\frac{\partial f(\mathbf{x})}{\partial x_i}$	$\boldsymbol{\nabla} f(\mathbf{x})$	"$= \frac{\mathrm{d}f(\mathbf{x})}{\mathrm{d}\mathbf{x}}$"	$\mathrm{d}f(\mathbf{x})$	$\mathrm{d}\mathbf{x}$
$\frac{\delta F[a]}{\delta a(x)}$	$\tilde{\nabla} F[a]$	"$= \frac{\delta F[a]}{\delta a}$"	$\delta F[a]$	δa

Differential relation	Linear approximation (for ϵ small enough)
$\mathrm{d}f(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x})\cdot\mathrm{d}\mathbf{x}$	$f(\mathbf{x}+\epsilon\hat{\mathbf{e}}_i) \approx f(\mathbf{x}) + \epsilon\frac{\partial f(\mathbf{x})}{\partial x_i}$
$\delta F[a] = \tilde{\nabla} F[a]\cdot\delta a$	$F[a+\epsilon\delta_x] \approx F[a] + \epsilon\frac{\delta F[a]}{\delta a(x)}$

Note: The actual motivation for the functional derivative may be that we want

$$\frac{\delta}{\delta a(x)} \int \mathrm{d}y\ J(y)\,a(y) = J(x).$$

The definition given above is one way to construct a transformation that has this property.

Now for some directional derivatives:

$$\begin{aligned}
\frac{\partial_n f(\mathbf{x})}{\partial_n x} &\equiv \lim_{\epsilon\to 0} \frac{f(\mathbf{x}+\epsilon\hat{\mathbf{n}}) - f(\mathbf{x})}{\epsilon} && \left(\text{known as } \tfrac{\partial f(\mathbf{x})}{\partial n}\right)\\
\frac{\delta_\lambda F[a]}{\delta_\lambda a} &\equiv \lim_{\epsilon\to 0} \frac{F[a+\epsilon\lambda] - F[a]}{\epsilon} && \left(\text{could also be known as } \tfrac{\delta F[a]}{\delta\lambda}\right)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}f(\mathbf{x}) &= \frac{\partial_n f(\mathbf{x})}{\partial_n x}\,\mathrm{d}\xi\\
&= \left\{\sum_{i=1}^{d}\left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right) n_i\right\}\mathrm{d}\xi \;=\; \left(\sum_{i=1}^{d}\left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right)\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{d} n_j\,\hat{\mathbf{e}}_j\right)\mathrm{d}\xi\\
&= \left\{\boldsymbol{\nabla} f(\mathbf{x})\right\}\cdot\hat{\mathbf{n}}\,\mathrm{d}\xi\\
\delta F[a] &= \frac{\delta_\lambda F[a]}{\delta_\lambda a}\,\mathrm{d}\xi\\
&= \left\{\int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\,\lambda(x)\right\}\mathrm{d}\xi \;=\; \left(\int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\,\delta_x\right)\cdot\left(\int \mathrm{d}y\ \lambda(y)\,\delta_y\right)\mathrm{d}\xi\\
&= \left\{\tilde{\nabla} F[a]\right\}\cdot\lambda\,\mathrm{d}\xi
\end{aligned}$$

Directional derivative	Directional derivative composition (Vector derivative projection)	"Direction" property	Comment
$\frac{\partial_n f(\mathbf{x})}{\partial_n x}$	$\frac{\partial_n f(\mathbf{x})}{\partial_n x} = \sum_i \left(\frac{\partial f(\mathbf{x})}{\partial x_i}\right) n_i = \boldsymbol{\nabla} f(\mathbf{x})\cdot\hat{\mathbf{n}}$	$\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = \sum_i {n_i}^2 = 1$	Length of $\hat{\mathbf{n}} = 1$
$\frac{\delta_\lambda F[a]}{\delta_\lambda a}$	$\frac{\delta_\lambda F[a]}{\delta_\lambda a} = \int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\lambda(x) = \tilde{\nabla} F[a]\cdot\lambda$	$\lambda\cdot\lambda = \int \mathrm{d}x\ \lambda^2(x) = ?$	$\delta_x\cdot\delta_y = \delta(x-y)$

Differential relation	Linear approximation (for ϵ small enough)	(for $\Delta\mathbf{x}$, λ small enough)
$\mathrm{d}f(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x})\cdot\hat{\mathbf{n}}\,\mathrm{d}\xi$	$f(\mathbf{x}+\epsilon\hat{\mathbf{n}}) \approx f(\mathbf{x}) + \epsilon\frac{\partial_n f(\mathbf{x})}{\partial_n x}$	$f(\mathbf{x}+\Delta\mathbf{x}) \approx f(\mathbf{x}) + \boldsymbol{\nabla} f(\mathbf{x})\cdot\Delta\mathbf{x}$
$\delta F[a] = \tilde{\nabla} F[a]\cdot\lambda\,\mathrm{d}\xi$	$F[a+\epsilon\lambda] \approx F[a] + \epsilon\frac{\delta_\lambda F[a]}{\delta_\lambda a}$	$F[a+\lambda] \approx F[a] + \tilde{\nabla} F[a]\cdot\lambda$

Is the following true? (Prove it.)

$$\begin{aligned}
\frac{\delta_\lambda F[a]}{\delta_\lambda a} &= \int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\,\lambda(x)\\
\frac{\delta_{\delta_x} F[a]}{\delta_{\delta_x} a} &= \int \mathrm{d}y\ \frac{\delta F[a]}{\delta a(y)}\,\delta(y-x) \;=\; \frac{\delta F[a]}{\delta a(x)} && \text{(At least this checks out.)}
\end{aligned}$$

Taylor series expansion[1] (to third order)
$f(\mathbf{x}+\Delta\mathbf{x}) = f(\mathbf{x}) + \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}\,\Delta x_i + \sum_{ij}\frac{\partial^2 f(\mathbf{x})}{\partial x_i\,\partial x_j}\Delta x_i\,\Delta x_j + \sum_{ijk}\frac{\partial^3 f(\mathbf{x})}{\partial x_i\,\partial x_j\,\partial x_k}\,\Delta x_i\,\Delta x_j\,\Delta x_k$
$F[a+\lambda] = F[a] + \int \mathrm{d}x\frac{\delta F[a]}{\delta a(x)}\lambda(x) + \int \mathrm{d}^2x\frac{\delta^2 F[a]}{\delta a(x_1)\,\delta a(x_2)}\lambda(x_1)\lambda(x_2) + \int \mathrm{d}^3x\frac{\delta^3 F[a]}{\delta a(x_1)\delta a(x_2)\delta a(x_3)}\lambda(x_1)\lambda(x_2)\lambda(x_3)$

[1]See the section on second order functional derivatives below.

Functional Integration and Measures

Volume integral	Differential product (Measure?) (Volume form?) (bad notation)	Recall
$\int \mathrm{d}^d x$	$\mathrm{d}^d x = \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_d = \prod_i \mathrm{d}x_i$ "$= \mathrm{d}\mathbf{x}$"	$\mathrm{d}\mathbf{x} = \sum_i \mathrm{d}x_i\, \hat{\mathbf{e}}_i$
$\int \mathcal{D}a$	$\mathcal{D}a = \prod_x \delta a(x) = \delta^\infty a$ "$= \delta a$"	$\delta a = \int \mathrm{d}x\ \delta a(x)\, \delta_x$

Functional Derivative Rules

Let C be a constant and let k be a variable-function independent of the variable-function a. Let F and G be functionals and f and g be functions.

$$\frac{\delta}{\delta a(x)} C = 0 \tag{1}$$

$$\frac{\delta}{\delta a(x)} F[k] = 0 \tag{2}$$

$$\frac{\delta}{\delta a(x)} \big(F[a] + G[a]\big) = \frac{\delta F[a]}{\delta a(x)} + \frac{\delta G[a]}{\delta a(x)} \tag{3}$$

$$\frac{\delta}{\delta a(x)} \big(F[a]\, G[a]\big) = \frac{\delta F[a]}{\delta a(x)}\ G[a] + F[a]\ \frac{\delta G[a]}{\delta a(x)} \tag{4}$$

$$\frac{\delta}{\delta a(x)} f(F[a]) = \left.\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right|_{z=F[a]} \frac{\delta F[a]}{\delta a(x)} \tag{5}$$

$$\frac{\delta}{\delta a(x)} f(F_1[a], \ldots, F_n[a]) = \sum_{i=1}^{n} \left.\frac{\partial f(\mathbf{z})}{\partial z_i}\right|_{z_i=F_i[a]} \frac{\delta F_i[a]}{\delta a(x)} \tag{6}$$

$$\frac{\delta}{\delta a(x)} g\big(a(y)\big) = \lim_{\epsilon\to 0} \frac{g\big(a(y) + \epsilon\delta(y-x)\big) - g\big(a(y)\big)}{\epsilon} \tag{7}$$

$$\frac{\delta}{\delta a(x)} a(y) = \delta(y-x) \tag{8}$$

$$\frac{\delta}{\delta a(x)} g\big(a(z_1), \ldots, a(z_n)\big) = \sum_{i=1}^{n} \left.\frac{\partial g(\mathbf{y})}{\partial y_i}\right|_{y_i=a(z_i)} \delta(z_i - x) \tag{9}$$

$$\frac{\delta}{\delta a(x)} \int \mathrm{d}z_1 \cdots \mathrm{d}z_n\ g\big(a(z_1), \ldots, a(z_n)\big) = \int \mathrm{d}z_1 \cdots \mathrm{d}z_n\ \frac{\delta}{\delta a(x)} g\big(a(z_1), \ldots, a(z_n)\big) \tag{10}$$

$$\frac{\delta}{\delta a(x)} a(y) = \delta(y-x) \tag{11}$$

$$\frac{\delta}{\delta a(x)} a^n(y) = n\, a^{n-1}(y)\, \delta(y-x) \tag{12}$$

$$\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ a^n(z) = n\, a^{n-1}(x) \tag{13}$$

$$\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ a^n(z) = \int \mathrm{d}z\ \frac{\delta}{\delta a(x)} a^n(z) \tag{14}$$

$$\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ a(z) = \int \mathrm{d}z\ \frac{\delta}{\delta a(x)} a(z) = 1 \tag{15}$$

$$
\begin{aligned}
\frac{\delta^2}{\delta a(x)^2}\, a(y) &= \frac{\delta}{\delta a(x)}\, \delta(y-x) \\
&= 0 \\
\frac{\delta^2}{\delta a(x)^2}\, a^2(y) &= 2\, \frac{\delta}{\delta a(x)}\, a(y)\, \delta(y-x) \\
&= 2 \lim_{\epsilon \to 0} \frac{\big(a(y) + \epsilon\delta(y-x)\big)\, \delta(y-x) - a(y)\, \delta(y-x)}{\epsilon} \\
&= 2\, \delta^2(y-x) \quad !
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta}{\delta a(x)}\, k(y)\, a(y) &= k(y)\, \delta(y-x) && (16) \\
\frac{\delta}{\delta a(x)}\, k(y)\, a^n(y) &= k(y)\, n\, a^{n-1}(y)\, \delta(y-x) && (17) \\
\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ k(z)\, a^n(z) &= k(x)\, n\, a^{n-1}(x) && (18) \\
\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ k(z)\, a^n(z) &= \int \mathrm{d}z\ \frac{\delta}{\delta a(x)}\, k(z)\, a^n(z) && (19) \\
\frac{\delta}{\delta a(x)} \int \mathrm{d}z\ k(z)\, a(z) &= \int \mathrm{d}z\ \frac{\delta}{\delta a(x)}\, k(z)\, a(z) \quad = \quad k(x) && (20)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta_\lambda}{\delta_\lambda a}\, a(y) &= \lambda(y) && (21) \\
\frac{\delta_\lambda}{\delta_\lambda a}\, a^n(y) &= n\, a^{n-1}(y)\, \lambda(y) && (22) \\
\frac{\delta_\lambda}{\delta_\lambda a} \int \mathrm{d}z\ a^n(z) &= \int \mathrm{d}z\ n\, a^{n-1}(z)\, \lambda(z) && (23) \\
\frac{\delta_\lambda}{\delta_\lambda a} \int \mathrm{d}z\ a^n(z) &= \int \mathrm{d}z\ \frac{\delta_\lambda}{\delta_\lambda a}\, a^n(z) && (24) \\
\frac{\delta_\lambda}{\delta_\lambda a} \int \mathrm{d}z\ a(z) &= \int \mathrm{d}z\ \frac{\delta_\lambda}{\delta_\lambda a}\, a(z) \quad = \quad \int \mathrm{d}z\ \lambda(z) && (25)
\end{aligned}
$$

$$
\frac{\delta_\lambda}{\delta_\lambda a}\, a^n(y) = \frac{\delta}{\delta a(y)} \int \mathrm{d}z\ \lambda(z)\, a^n(z) \qquad (26)
$$

Examples of composition and product rules:

$$
\begin{aligned}
\frac{\delta}{\delta a(x)}\, \frac{1}{F[a]} &= -\frac{1}{F[a]^2}\, \frac{\delta F[a]}{\delta a(x)} \\
\frac{\delta}{\delta a(x)}\, a^m(y)\, a^n(z) &= m\, a^{m-1}(y)\, a^n(z)\, \delta(y-x) + a^m(y)\, n\, a^{n-1}(z)\, \delta(z-x)
\end{aligned}
$$

If we define

$$\frac{\delta_\lambda F[a]}{\delta_\lambda a(x)} \equiv \lim_{\epsilon\to 0} \frac{F[a+\epsilon\lambda\delta_x]-F[a]}{\epsilon}$$

then (compare the following Eqns. to Eqns. 11 - 15 and Eqns. 16 - 20)

$$\begin{aligned}
\frac{\delta_\lambda}{\delta_\lambda a(x)}\, a(y) &= \lambda(y)\,\delta(y-x) && (27)\\
\frac{\delta_\lambda}{\delta_\lambda a(x)}\, a^n(y) &= \lambda(y)\, n\, a^{n-1}(y)\,\delta(y-x) && (28)\\
\frac{\delta_\lambda}{\delta_\lambda a(x)} \int \mathrm{d}z\ a^n(z) &= \lambda(x)\, n\, a^{n-1}(x) && (29)\\
\frac{\delta_\lambda}{\delta_\lambda a(x)} \int \mathrm{d}z\ a^n(z) &= \int \mathrm{d}z\ \frac{\delta}{\delta a(x)}\,\lambda(z)\, a^n(z) && (30)\\
\frac{\delta_\lambda}{\delta_\lambda a(x)} \int \mathrm{d}z\ a(z) &= \int \mathrm{d}z\ \frac{\delta}{\delta a(x)}\,\lambda(z)\, a(z) \quad = \quad \lambda(x) && (31)
\end{aligned}$$

It seems that

$$\frac{\delta_\lambda F[a]}{\delta_\lambda a(x)} = \frac{\delta F[a]}{\delta a(x)}\,\lambda(x)$$

and

$$\frac{\delta_\lambda F[a]}{\delta_\lambda a} = \int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\,\lambda(x) \tag{32}$$

so it also seems that

$$\frac{\delta_\lambda F[a]}{\delta_\lambda a} = \int \mathrm{d}x\ \frac{\delta_\lambda F[a]}{\delta_\lambda a(x)}$$

Is this true in general, for any functional $F[a]$?

Let

$$F_{kn}[a] = \int \mathrm{d}z\ k(z)\, a^n(z) \qquad \text{and} \qquad F_{1n}[a] = \int \mathrm{d}z\ a^n(z),$$

$$\begin{aligned}
\frac{\delta_\lambda F_{1n}[a]}{\delta_\lambda a} &= \int \mathrm{d}x\ \frac{\delta_\lambda F_{1n}[a]}{\delta_\lambda a(x)}\\
\frac{\delta_\lambda F_{1n}[a]}{\delta_\lambda a(x)} &= \frac{\delta F_{\lambda n}[a]}{\delta a(x)}
\end{aligned}$$

Derivative Rule Proofs and Semi-Proofs

(1) Constants. The functional derivative of a constant (functional) is zero. A constant C can be seen as a functional that always returns C ($C[a] = C$), and

$$\frac{\delta C}{\delta a(x)} \;=\; \lim_{\epsilon\to 0}\frac{C[a+\epsilon\delta_x]-C[a]}{\epsilon} \;=\; \lim_{\epsilon\to 0}\frac{C-C}{\epsilon} \;=\; \lim_{\epsilon\to 0} 0 \;=\; 0$$

(2) Functional independence. The functional derivative operator $\frac{\delta}{\delta a(x)}$ can be seen as the instruction "Take the functional that is being differentiated and wherever you see the function a (or wherever it is lurking, if another function depends on a), substitute a with $a + \epsilon\delta_x$, then subtract the original functional, divide the whole thing by ϵ, and take the limit as ϵ goes to zero."

For the case of the functional $F[k]$ where k is independent of a, there is no place to substitute a with $a + \epsilon\delta_x$, so we immediately get a difference of zero, and the functional derivative is zero:

$$\frac{\delta}{\delta a(x)}\,F[k] = \lim_{\epsilon\to 0}\frac{F[k]-F[k]}{\epsilon} = \lim_{\epsilon\to 0} 0 = 0$$

(3) Sum rule. (Linearity of the funcitonal derivative)

$$\begin{aligned}
\frac{\delta}{\delta a(x)}\big(F[a]+G[a]\big) &= \lim_{\epsilon\to 0}\frac{\big(F[a+\epsilon\delta_x]+G[a+\epsilon\delta_x]\big)-\big(F[a]+G[a]\big)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{F[a+\epsilon\delta_x]-F[a]}{\epsilon}+\lim_{\epsilon\to 0}\frac{G[a+\epsilon\delta_x]-G[a]}{\epsilon}\\
&= \frac{\delta F[a]}{\delta a(x)}+\frac{\delta G[a]}{\delta a(x)}
\end{aligned}$$

(4) Product rule.

$$\begin{aligned}
\frac{\delta}{\delta a(x)}\big(F[a]\,G[a]\big) &= \lim_{\epsilon\to 0}\frac{F[a+\epsilon\delta_x]\,G[a+\epsilon\delta_x]-F[a]\,G[a]}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{\Big(F[a]+\epsilon\frac{\delta F[a]}{\delta a(x)}\Big)\Big(G[a]+\epsilon\frac{\delta G[a]}{\delta a(x)}\Big)-F[a]\,G[a]}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{F[a]\,G[a]+\epsilon\frac{\delta F[a]}{\delta a(x)}\,G[a]+\epsilon\,F[a]\,\frac{\delta G[a]}{\delta a(x)}+\epsilon^2\frac{\delta F[a]}{\delta a(x)}\,\frac{\delta G[a]}{\delta a(x)}-F[a]\,G[a]}{\epsilon}\\
&= \frac{\delta F[a]}{\delta a(x)}\,G[a]+F[a]\,\frac{\delta G[a]}{\delta a(x)}
\end{aligned}$$

(5) Composition rule. ("Chain rule")

We can compose a function f from a field of scalars $\mathcal{K}$ to $\mathcal{K}$, $f : \mathcal{K} \to \mathcal{K}$, with a functional F from a function space $\mathcal{A}$ to the field $\mathcal{K}$, $F : \mathcal{A} \to \mathcal{K}$, yielding the composite functional $f \circ F$ from $\mathcal{A}$ to $\mathcal{K}$, $f \circ F : \mathcal{A} \to \mathcal{K}$, defined by $(f \circ F)[a] = f(F[a])$.

$$\begin{aligned}
\frac{\delta}{\delta a(x)}\,f(F[a]) &= \lim_{\epsilon\to 0}\frac{f(F[a+\epsilon\delta_x])-f(F[a])}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f\Big(F[a]+\epsilon\frac{\delta F[a]}{\delta a(x)}\Big)-f(F[a])}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f(F[a])+\epsilon\frac{\delta F[a]}{\delta a(x)}\,\frac{\mathrm{d}f(z)}{\mathrm{d}z}\Big|_{z=F[a]}-f(F[a])}{\epsilon}\\
&= \frac{\mathrm{d}f(z)}{\mathrm{d}z}\bigg|_{z=F[a]}\,\frac{\delta F[a]}{\delta a(x)}
\end{aligned}$$

(8) The function $a(y)$ is a functional in the sense that it can be expressed as $F_y[a] = a(y)$, or $F_y[a] = \int \mathrm{d}\xi\ a(\xi)\,\delta(\xi - y) = a \cdot \delta_y = a(y)$; the functional F_y is given a function a and returns a number $a(y)$, the function's value at position y. Thus we may take a functional derivative of $a(y)$:

$$
\begin{aligned}
\frac{\delta a(y)}{\delta a(x)} &= \frac{\delta F_y[a]}{\delta a(x)} = \lim_{\epsilon \to 0} \frac{F_y[a + \epsilon\delta_x] - F_y[a]}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\int \mathrm{d}\xi\ \{\cancel{a(\xi)} + \epsilon\delta(\xi - x)\}\,\delta(\xi - y) - \int \mathrm{d}\xi\ \cancel{a(\xi)}\,\delta(\xi - y)}{\epsilon} \\
&= \int \mathrm{d}\xi \delta(\xi - x)\,\delta(\xi - y) = \delta(y - x)
\end{aligned}
$$

or, simply,

$$
\begin{aligned}
\frac{\delta a(y)}{\delta a(x)} &= \lim_{\epsilon \to 0} \frac{a(y) + \epsilon\delta(y - x) - a(y)}{\epsilon} \\
&= \delta(y - x)
\end{aligned}
$$

(9) Maybe show the integral version first, so that this version can be justified. (However, in the integral version we'll have to deal with polynomials and functions of delta functions/distributions.)

Using

$$
\begin{aligned}
g(y_1 + \epsilon_1, \dots, y_n + \epsilon_n) &= g(\mathbf{y} + \boldsymbol{\epsilon}) \\
&= g(\mathbf{y}) + \boldsymbol{\nabla}_{\mathbf{y}}\, g(\mathbf{y}) \cdot \boldsymbol{\epsilon} \\
&= g(y_1, \dots, y_n) + \sum_i \frac{\partial g}{\partial y_i}(y_1, \dots, y_n)\, w_i
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\delta}{\delta a(x)}\, g\big(a(z_1), \dots, a(z_n)\big) &= \lim_{\epsilon \to 0} \frac{g\big(a(z_1) + \epsilon\delta(z_1 - x), \dots, a(z_n) + \epsilon\delta(z_n - x)\big) - g\big(a(z_1), \dots, a(z_n)\big)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{g\big(a(z_1) + \epsilon\delta(z_1 - x), \dots, a(z_n) + \epsilon\delta(z_n - x)\big) - g\big(a(z_1), \dots, a(z_n)\big)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\sum_i \left. \frac{\partial g(y_1, \dots, y_n)}{\partial y_i} \right|_{y_i = a(z_i)} \epsilon\,\delta(z_i - x)}{\epsilon} \\
&= \sum_{i=1}^{n} \left. \frac{\partial g(\mathbf{y})}{\partial y_i} \right|_{y_i = a(z_i)} \delta(z_i - x)
\end{aligned}
$$

Integral version:

$$
\frac{\delta}{\delta a(x)} \int \mathrm{d}z_1 \cdots \mathrm{d}z_n\ g\big(a(z_1), \dots, a(z_n)\big) = \dots
$$

(11) The function $a(y)$ is a functional in the sense that it can be expressed as $F_y[a] = a(y)$, or $F_y[a] = \int \mathrm{d}\xi\ a(\xi)\,\delta(\xi - y) = a \cdot \delta_y = a(y)$; the functional F_y is given a function a and returns a number $a(y)$, the function's value at position y. Thus we may take a functional derivative of $a(y)$:

$$
\begin{aligned}
\frac{\delta a(y)}{\delta a(x)} &= \frac{\delta F_y[a]}{\delta a(x)} = \lim_{\epsilon \to 0} \frac{F_y[a + \epsilon\delta_x] - F_y[a]}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\int \mathrm{d}\xi\ \{\cancel{a(\xi)} + \epsilon\delta(\xi - x)\}\,\delta(\xi - y) - \int \mathrm{d}\xi\ \cancel{a(\xi)}\,\delta(\xi - y)}{\epsilon} \\
&= \int \mathrm{d}\xi \delta(\xi - x)\,\delta(\xi - y) = \delta(y - x)
\end{aligned}
$$

or, simply,

$$\begin{aligned}\frac{\delta a(y)}{\delta a(x)} &= \lim_{\epsilon\to 0}\frac{a(y)+\epsilon\delta(y-x)-a(y)}{\epsilon}\\ &= \delta(y-x)\end{aligned}$$

(12) The function $a^n(y)$ can also be written as $G_y[a] = \int \mathrm{d}\xi\ a^n(\xi)\,\delta(\xi - y)$, but as we saw above, we don't need to analyze the derivative in terms of this explicit functional $G_y[a]$.

$$\begin{aligned}\frac{\delta}{\delta a(x)}\,a^n(y) &= \lim_{\epsilon\to 0}\frac{\big\{a(y)+\epsilon\delta(y-x)\big\}^n - a^n(y)}{\epsilon}\\ &= \text{write out binomial expansion and cancel terms}\\ &= n\,a^{n-1}(y)\,\delta(y-x)\end{aligned}$$

(32) If

$$\begin{aligned}F[a+\epsilon\lambda] &= F\left[a+\epsilon\int \mathrm{d}x\ \lambda(x)\,\delta_x\right]\\ &= F[a]+\epsilon\int \mathrm{d}x\ \lambda(x)\,\tfrac{\delta F[a]}{\delta a(x)}\end{aligned}$$

like

$$\begin{aligned}f[\mathbf{x}+\mathrm{d}\mathbf{x}] &= f(\mathbf{x}+\sum_i \mathrm{d}x_i\,\hat{\mathbf{e}}_i)\\ &= f(\mathbf{x})+\sum \mathrm{d}x_i\,\tfrac{\partial f(\mathbf{x})}{\partial x_i}\\ &= f(\mathbf{x})+\mathrm{d}\mathbf{x}\cdot\boldsymbol{\nabla} f(\mathbf{x})\end{aligned}$$

then

$$\begin{aligned}\frac{\delta_\lambda F[a]}{\delta_\lambda a} &= \lim_{\epsilon\to 0}\frac{F[a+\epsilon\lambda]-F[a]}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{F[a+\epsilon\int \mathrm{d}x\ \lambda(x)\,\delta_x]-F[a]}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{F[a]+\epsilon\int \mathrm{d}x\ \lambda(x)\,\frac{\delta F[a]}{\delta a(x)}-F[a]}{\epsilon}\\ &= \int \mathrm{d}x\ \frac{\delta F[a]}{\delta a(x)}\,\lambda(x)\end{aligned}$$

Hatfield Exercises

Chapter 9 Exercises

Exercise 2.

Compute

$$\int \mathrm{d}x \ \frac{\delta^2}{\delta\varphi(x)^2} \exp\left(-\int \mathrm{d}y\,\mathrm{d}z\ \varphi^2(y)\,g(y,z)\,\varphi(z) + \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\ \varphi(y)\,f(y,z)\,\varphi(z)\,h(z,s)\,\varphi(s)\right)$$

Solution 2.

Let

$$F[\varphi] \quad = \quad -\int \mathrm{d}y\,\mathrm{d}z\ \varphi^2(y)\,g(y,z)\,\varphi(z) + \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\ \varphi(y)\,f(y,z)\,\varphi(z)\,h(z,s)\,\varphi(s)$$

So we want to compute

$$\int \mathrm{d}x \ \frac{\delta^2}{\delta\varphi(x)^2}\, e^{F[\varphi]}$$

$$\begin{aligned}
\frac{\delta}{\delta\varphi(x)}\, e^{F[\varphi]} &= \left.\frac{\mathrm{d}(e^t)}{\mathrm{d}t}\right|_{t=F[\varphi]} \frac{\delta F[\varphi]}{\delta\varphi(x)} \quad = \quad \left. e^t\right|_{t=F[\varphi]} \frac{\delta F[\varphi]}{\delta\varphi(x)} \quad = \quad e^{F[\varphi]}\,\frac{\delta F[\varphi]}{\delta\varphi(x)} \\
\frac{\delta^2}{\delta\varphi(x)^2}\, e^{F[\varphi]} &= \frac{\delta}{\delta\varphi(x)}\left(e^{F[\varphi]}\,\frac{\delta F[\varphi]}{\delta\varphi(x)}\right) \\
&= \left(\frac{\delta}{\delta\varphi(x)}\, e^{F[\varphi]}\right)\frac{\delta F[\varphi]}{\delta\varphi(x)} + e^{F[\varphi]}\,\frac{\delta^2 F[\varphi]}{\delta\varphi(x)^2} \\
&= e^{F[\varphi]}\left[\left(\frac{\delta F[\varphi]}{\delta\varphi(x)}\right)^2 + \frac{\delta^2 F[\varphi]}{\delta\varphi(x)^2}\right] \\
\frac{\delta F[\varphi]}{\delta\varphi(x)} &= -\int \mathrm{d}y\,\mathrm{d}z\ 2\varphi(y)\,\delta(y-x)\,g(y,z)\,\varphi(z) - \int \mathrm{d}y\,\mathrm{d}z\ \varphi^2(y)\,g(y,z)\,\delta(z-x) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\ \delta(y-x)\,f(y,z)\,\varphi(z)\,h(z,s)\,\varphi(s) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\ \varphi(y)\,f(y,z)\,\delta(z-x)\,h(z,s)\,\varphi(s) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\ \varphi(y)\,f(y,z)\,\varphi(z)\,h(z,s)\,\delta(s-x) \\
&= -2\varphi(x)\int \mathrm{d}z\ g(x,z)\,\varphi(z) - \int \mathrm{d}y\ \varphi^2(y)\,g(y,x) \\
&\quad + \int \mathrm{d}z\,\mathrm{d}s\ f(x,z)\,\varphi(z)\,h(z,s)\,\varphi(s) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}s\ \varphi(y)\,f(y,x)\,h(x,s)\,\varphi(s) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}z\ \varphi(y)\,f(y,z)\,\varphi(z)\,h(z,x)
\end{aligned}$$

$$\frac{\delta^2 F[\varphi]}{\delta\varphi(x)^2} \quad = \quad \cdots \text{ uh oh, this looks to be undefined}\ldots$$

I think

$$\frac{\delta^2}{\delta\varphi(x)^2}$$

is undefined. I will replace it with

$$\frac{\delta}{\delta\varphi(x_2)}\,\frac{\delta}{\delta\varphi(x_1)}$$

(and should I replace

$$\int \mathrm{d}x \quad \text{with} \quad \int \mathrm{d}x_2\,\mathrm{d}x_2 \quad ? \;)$$

so

$$\begin{aligned}
\frac{\delta}{\delta\varphi(x_2)}\,\frac{\delta F[\varphi]}{\delta\varphi(x_1)} &= -2\,\delta(x_1-x_2)\int \mathrm{d}z\; g(x_1,z)\,\varphi(z) - 2\int \mathrm{d}y\;\varphi(y)\,\delta(y-x_2)\,g(y,x_1) \\
&\quad + \int \mathrm{d}z\,\mathrm{d}s\; f(x_1,z)\,\delta(z-x_2)\,h(z,s)\,\varphi(s) + \int \mathrm{d}z\,\mathrm{d}s\; f(x_1,z)\,\varphi(z)\,h(z,s)\,\delta(s-x_2) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}s\; \delta(y-x_2)\,f(y,x_1)\,h(x_1,s)\,\varphi(s) + \int \mathrm{d}y\,\mathrm{d}s\; \varphi(y)\,f(y,x_1)\,h(x_1,s)\,\delta(s-x_2) \\
&\quad + \int \mathrm{d}y\,\mathrm{d}z\; \delta(y-x_2)\,f(y,z)\,\varphi(z)\,h(z,x_1) + \int \mathrm{d}y\,\mathrm{d}z\; \varphi(y)\,f(y,z)\,\delta(z-x_2)\,h(z,x_1) \\
&= -2\,\delta(x_1-x_2)\int \mathrm{d}z\; g(x_1,z)\,\varphi(z) - 2\,\varphi(x_2)\,g(x_2,x_1) \\
&\quad + \int \mathrm{d}s\; f(x_1,x_2)\,h(x_2,s)\,\varphi(s) + \int \mathrm{d}z\; f(x_1,z)\,\varphi(z)\,h(z,x_2) \\
&\quad + \int \mathrm{d}s\; f(x_2,x_1)\,h(x_1,s)\,\varphi(s) + \int \mathrm{d}y\; \varphi(y)\,f(y,x_1)\,h(x_1,x_2) \\
&\quad + \int \mathrm{d}z\; f(x_2,z)\,\varphi(z)\,h(z,x_1) + \int \mathrm{d}y\; \varphi(y)\,f(y,x_2)\,h(x_2,x_1) \\
&= -2\,\delta(x_1-x_2)\int \mathrm{d}z\; g(x_1,z)\,\varphi(z) - 2\,\varphi(x_2)\,g(x_2,x_1) \\
&\quad + \int \mathrm{d}s\; \varphi(s)\,\big[f(x_1,x_2)\,h(x_2,s) + f(x_2,x_1)\,h(x_1,s)\big] \\
&\quad + \int \mathrm{d}y\; \varphi(y)\,\big[f(y,x_2)\,h(x_2,x_1) + f(y,x_1)\,h(x_1,x_2)\big] \\
&\quad + \int \mathrm{d}z\; \varphi(z)\,\big[f(x_2,z)\,h(z,x_1) + f(x_1,z)\,h(z,x_2)\big] \\
&= -2\,\varphi(x_2)\,g(x_2,x_1) - 2\,\delta(x_1-x_2)\int \mathrm{d}\xi\; g(x_1,\xi)\,\varphi(\xi) \\
&\quad + \int \mathrm{d}\xi\; \varphi(\xi)\,\big[f(x_1,x_2)\,h(x_2,\xi) + f(x_2,x_1)\,h(x_1,\xi)\big] \\
&\quad + \int \mathrm{d}\xi\; \varphi(\xi)\,\big[f(\xi,x_2)\,h(x_2,x_1) + f(\xi,x_1)\,h(x_1,x_2)\big] \\
&\quad + \int \mathrm{d}\xi\; \varphi(\xi)\,\big[f(x_2,\xi)\,h(\xi,x_1) + f(x_1,\xi)\,h(\xi,x_2)\big]
\end{aligned}$$

Thus, the answer is

$$\begin{aligned}
\int \mathrm{d}x \ \frac{\delta^2}{\delta\varphi(x)^2}\, e^{F[\varphi]} &= \int \mathrm{d}x \ e^{F[\varphi]} \left[\left(\frac{\delta F[\varphi]}{\delta\varphi(x)} \right)^2 + \frac{\delta^2 F[\varphi]}{\delta\varphi(x)^2} \right] \\
&\to \int \mathrm{d}x_1\, \mathrm{d}x_2 \ \frac{\delta}{\delta\varphi(x_1)} \frac{\delta}{\delta\varphi(x_2)}\, e^{F[\varphi]} \\
&= \int \mathrm{d}x_1\, \mathrm{d}x_2 \ e^{F[\varphi]} \left[\left(\frac{\delta F[\varphi]}{\delta\varphi(x_1)} \right) \left(\frac{\delta F[\varphi]}{\delta\varphi(x_2)} \right) + \frac{\delta}{\delta\varphi(x_1)} \frac{\delta}{\delta\varphi(x_2)}\, F[\varphi] \right]
\end{aligned}$$

$$\begin{aligned}
= \ & \int \mathrm{d}x_1\, \mathrm{d}x_2 \ \exp\left(-\int \mathrm{d}y\, \mathrm{d}z \ \varphi^2(y)\, g(y,z)\, \varphi(z) + \int \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}s \ \varphi(y)\, f(y,z)\, \varphi(z)\, h(z,s)\, \varphi(s) \right) \\
& \times \Bigg[\Bigg(-2\varphi(x_1) \int \mathrm{d}z \ g(x_1,z)\, \varphi(z) - \int \mathrm{d}y \ \varphi^2(y)\, g(y,x_1) + \int \mathrm{d}z\, \mathrm{d}s \ f(x_1,z)\, \varphi(z)\, h(z,s)\, \varphi(s) \\
& \qquad + \int \mathrm{d}y\, \mathrm{d}s \ \varphi(y)\, f(y,x_1)\, h(x_1,s)\, \varphi(s) + \int \mathrm{d}y\, \mathrm{d}z \ \varphi(y)\, f(y,z)\, \varphi(z)\, h(z,x_1) \Bigg) \\
& \quad \times \Bigg(-2\varphi(x_2) \int \mathrm{d}z \ g(x_2,z)\, \varphi(z) - \int \mathrm{d}y \ \varphi^2(y)\, g(y,x_2) + \int \mathrm{d}z\, \mathrm{d}s \ f(x_2,z)\, \varphi(z)\, h(z,s)\, \varphi(s) \\
& \qquad + \int \mathrm{d}y\, \mathrm{d}s \ \varphi(y)\, f(y,x_2)\, h(x_2,s)\, \varphi(s) + \int \mathrm{d}y\, \mathrm{d}z \ \varphi(y)\, f(y,z)\, \varphi(z)\, h(z,x_2) \Bigg) \\
& \quad -2\,\varphi(x_2)\, g(x_2,x_1) - 2\,\delta(x_1 - x_2) \int \mathrm{d}\xi \ g(x_1,\xi)\, \varphi(\xi) + \int \mathrm{d}\xi \ \varphi(\xi) \left[f(x_1,x_2)\, h(x_2,\xi) + f(x_2,x_1)\, h(x_1,\xi) \right] \\
& \quad + \int \mathrm{d}\xi \ \varphi(\xi) \left[f(\xi,x_2)\, h(x_2,x_1) + f(\xi,x_1)\, h(x_1,x_2) \right] + \int \mathrm{d}\xi \ \varphi(\xi) \left[f(x_2,\xi)\, h(\xi,x_1) + f(x_1,\xi)\, h(\xi,x_2) \right] \Bigg]
\end{aligned}$$

Solving a Functional Differential Equation (from Hatfield pg 183)

We'd like to solve the functional differential equation (Hatfield Equation (9.16))

$$\int \mathrm{d}y \ \frac{\delta}{\delta a(y)}\, F[a] = -\mu\, F[a]$$

for the functional $F[a]$. We assume that $F[a]$ is separable; that is,

$$F[a] = \eta \prod_x \mathcal{F}\big(x, a(x)\big)\,,$$

where η is a constant and $\mathcal{F}$ is some two-argument function. If one wants to know what this means more rigorously, we can use the properties of exponents to write

$$F[a] = \eta \prod_x \mathcal{F}\big(x, a(x)\big) = \eta \prod_x e^{\ln \mathcal{F}(x, a(x))} = \eta\, e^{\int \mathrm{d}x\, \ln \mathcal{F}(x, a(x))},$$

but we won't use this yet.

With this assumption we have

$$\begin{aligned}
\frac{\delta}{\delta a(y)}\, F[a] &= \eta\, \frac{\delta}{\delta a(y)} \left(\prod_x \mathcal{F}\big(x, a(x)\big) \right) \\
&= \eta \lim_{\epsilon \to 0} \frac{\left(\prod_{x \neq y} \mathcal{F}\big(x, a(x)\big)\right) \mathcal{F}\big(y, a(y)+\epsilon\big) - \prod_x \mathcal{F}\big(x, a(x)\big)}{\epsilon} \\
&= \eta\, \frac{\prod_x \mathcal{F}\big(x, a(x)\big)}{\mathcal{F}\big(y, a(y)\big)} \left[\lim_{\epsilon \to 0} \frac{\mathcal{F}\big(y, a(y)+\epsilon\big) - \mathcal{F}\big(y, a(y)\big)}{\epsilon} \right] \\
&= F[a]\, \left. \frac{\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{F}(y,z)}{\mathcal{F}(y,z)} \right|_{z=a(y)} \\
&= F[a] \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)}
\end{aligned}$$

We should get the same thing using our more rigorous formula (and the rules for functional derivatives)

$$\begin{aligned}
\frac{\delta}{\delta a(y)}\, F[a] &= \eta\, \frac{\delta}{\delta a(y)} \left(e^{\int \mathrm{d}x\, \ln \mathcal{F}\left(x, a(x)\right)} \right) \\
&= \eta\, e^{\int \mathrm{d}x\, \ln \mathcal{F}\left(x, a(x)\right)}\, \frac{\delta}{\delta a(y)} \left(\int \mathrm{d}x\, \ln \mathcal{F}\big(x, a(x)\big) \right) \\
&= F[a] \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)}
\end{aligned}$$

and we do. Thus, our differential equation becomes

$$\int \mathrm{d}y\;\, F[a] \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)} = -\mu\, F[a]$$

$$\cancel{F[a]} \int \mathrm{d}y \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)} = -\mu\, \cancel{F[a]}$$

$$\int \mathrm{d}y \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)} = -\mu$$

We now make another simplifying assumption that $\mathcal{F}$ has an exponential form:

$$\mathcal{F}\big(y, a(y)\big) = e^{-f(y)\, a(y)},$$

so $\quad F[a] = \eta\, e^{\int \mathrm{d}x\, f(x)\, a(x)} \quad$ and $\quad \left[\tfrac{\mathrm{d}}{\mathrm{d}z} \ln \mathcal{F}(y,z) \right]_{z=a(y)} = \tfrac{\mathrm{d}}{\mathrm{d}z}\big(-f(y)\, z\big)\Big|_{z=a(y)} = -f(y)$

and we have

$$-\int \mathrm{d}y\;\, f(y) = -\mu.$$

Therefore, the functional differential equation

$$\int \mathrm{d}y\;\, \frac{\delta}{\delta a(y)}\, F[a] = -\mu\, F[a]$$

has as a solution

$$\boxed{F[a] = \eta\, e^{\int \mathrm{d}x\, f(x)\, a(x)} \qquad \text{where } f \text{ must satisfy} \qquad \int \mathrm{d}y\;\, f(y) = \mu}$$

Second Order Functional Derivatives

Taylor Expansion to Second Order

I think that one appropriate way of looking at the concept of the second order functional derivative is to follow the analogy with functions of finite-dimensional vectors, looking at the Taylor expansion to second order:

$$
\begin{aligned}
f(x+\epsilon) &= f(x) + \epsilon\, f'(x) + \frac{1}{2}\epsilon^2\, f''(x) \\
f(\mathbf{x}+\Delta\mathbf{x}) &= f(\mathbf{x}) + \boldsymbol{\nabla} f(\mathbf{x}) \cdot \Delta\mathbf{x} + \\
f(\mathbf{x}+\Delta\mathbf{x}) &= f(\mathbf{x}) + \sum_i \frac{\partial f(\mathbf{x})}{\partial x_i}\, \Delta x_i + \sum_{ij} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\, \Delta x_i\, \Delta x_j \\
F[a+\lambda] &= F[a] + \int \mathrm{d}x\; \frac{\delta F[a]}{\delta a(x)}\, \lambda(x) + \int \mathrm{d}x_1\, \mathrm{d}x_2\; \frac{\delta^2 F[a]}{\delta a(x_1)\, \delta a(x_2)}\, \lambda(x_1)\, \lambda(x_2)
\end{aligned}
$$

Thus we shouldn't care so much about

$$\frac{\delta^2}{\delta a(x)^2}\,.$$

What really matters is

$$\frac{\delta^2}{\delta a(x_1)\, \delta a(x_2)}$$

because it seems that we only really care about functional derivatives when they're in integrals (and because $\delta^2/\delta a(x)^2$ yields nonsense, naïvely at least).

Another Idea

There is another possibly appropriate way of looking at the concept of the second order functional derivative. My friend Matt Mahoney suggested that instead of using a Dirac delta, one could use a smooth (but narrow) bump function, somewhat like the delta function, to calculate the second-order dependence of the change in the functional on the variation in the function.

Attempt at an Explicit Formula for the Second Order Functional Derivative

This procedure for defining $\delta^2/\delta a(x)^2$ was proposed by my friend Matt Mahoney, but it turns out to be equivalent to the naïve two-fold application of the functional derivative:

$$
\begin{aligned}
\frac{\partial^2 f(\mathbf{x})}{\partial {x_i}^2} &\equiv \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}(\mathbf{x})\right) \\
&\equiv \lim_{b\to 0} \frac{\frac{\partial f}{\partial x_i}(\mathbf{x}+b\hat{\mathbf{e}}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x})}{b} \\
&\equiv \lim_{b\to 0} \frac{\left(\lim_{a\to 0} \frac{f(\mathbf{x}+b\hat{\mathbf{e}}_i+a\hat{\mathbf{e}}_i) - f(\mathbf{x}+b\hat{\mathbf{e}}_i)}{a}\right) - \left(\lim_{a\to 0} \frac{f(\mathbf{x}+a\hat{\mathbf{e}}_i) - f(\mathbf{x})}{a}\right)}{b} \\
&= \lim_{b\to 0} \lim_{a\to 0} \frac{f(\mathbf{x}+b\hat{\mathbf{e}}_i+a\hat{\mathbf{e}}_i) - f(\mathbf{x}+b\hat{\mathbf{e}}_i) - f(\mathbf{x}+a\hat{\mathbf{e}}_i) + f(\mathbf{x})}{ab} \\
&= \lim_{\epsilon\to 0} \frac{f(\mathbf{x}+2\epsilon\hat{\mathbf{e}}_i) - 2\, f(\mathbf{x}+\epsilon\hat{\mathbf{e}}_i) + f(\mathbf{x})}{\epsilon^2} \\
\frac{\delta^2 F[a]}{\delta a(x)^2} &\equiv \lim_{\epsilon\to 0} \frac{F[a+2\epsilon\delta_x] - 2\, F[a+\epsilon\delta_x] + F[a]}{\epsilon^2}
\end{aligned}
$$

That means

$$\begin{aligned}
\frac{\delta^2}{\delta a(x)^2}\, a^2(y) &= \lim_{\epsilon\to 0} \frac{\big(a(y)+2\epsilon\delta(x-y)\big)^2 - 2\,\big(a(y)+\epsilon\delta(x-y)\big)^2 + a^2(y)}{\epsilon^2} \\
&= \lim_{\epsilon\to 0} \frac{a^2(y) + 4\epsilon\, a(y)\,\delta(x-y) + 4\epsilon^2\delta^2(x-y) - 2\,a^2(y) - 4\epsilon\, a(y)\,\delta(x-y) - 2\epsilon^2\delta^2(x-y) + a^2(y)}{\epsilon^2} \\
&= \lim_{\epsilon\to 0} \frac{2\epsilon^2\delta^2(x-y)}{\epsilon^2} \\
&= 2\,\delta^2(x-y)
\end{aligned}$$

This is just what we get when we naïvely take the second functional derivative:

$$\begin{aligned}
\frac{\delta^2}{\delta a(x)^2}\, a^2(y) &= \frac{\delta}{\delta a(x)}\left(\frac{\delta}{\delta a(x)}\, a^2(y)\right) \\
&= \frac{\delta}{\delta a(x)}\,\big(2\,a(y)\,\delta(x-y)\big) \\
&= 2\,\delta(x-y)\,\frac{\delta}{\delta a(x)}\,\big(a(y)\big) \\
&= 2\,\delta(x-y)\;\delta(x-y) \\
&= 2\,\delta^2(x-y)
\end{aligned}$$

But as far as I know $\delta^2(x-y)$ is undefined and nonsense.

Another Question

Is this true?

$$\lim_{\epsilon\to 0}\int_{x-\epsilon}^{x+\epsilon}\int_{x-\epsilon}^{x+\epsilon} \mathrm{d}x_1\,\mathrm{d}x_2\,\frac{\delta^2 F[a]}{\delta a(x_1)\,\delta a(x_2)} \stackrel{?}{=} \frac{\delta^2 F[a]}{\delta a(x)^2}$$

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

- Chapter 9: Functional Calculus