Week 9 Lecture: Concepts of Quantum Field Theory (QFT)

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An Intuitive Picture of the QFT Field

This Week's Questions/Goals

- Graph a QFT state in the position+field-coordinate representation (as opposed to the position+field-momentum, momentum+field-coordinate, or momentum+field-momentum representations).
- Show the connection between a membrane/sheet deviation-function and a QFT wavefunction. (Use Hatfield's example function for the ground state, etc.)
- Derive the QFT ground state using the annihilation operator, as is done in the usual quantum harmonic oscillator problem.
- Keep working on the complete derivation for the real Klein-Gordon QFT field. (Lorentz-invariant measure, etc.)
- Keep working on the general solution (and raising/lowering operators) of a QFT field for various equations of motion. (What is the Hamiltonian in

$$\hat{H} \,\|\, \Psi \,\rangle\!\rangle = \hat{E} \,\|\, \Psi \,\rangle\!\rangle \,,$$

  and how does one time-propagate a state?)
- (Note that these fields are free solutions, meaning that one would not expect to find a stationary particle – for long, at least – since all the momentum-modes would propagate onward at different speeds.)
- Idea (related to a question last week):
  - The "raising operator" for the classical Klein-Gordon field is a distributed force that (immediately, almost instantaneously) creates a plane-waveform of a particular frequency. This is different from the quantum raising operator because it's magnitude is arbitrary and we can explicitly see that it takes some time to operate.
  - Perhaps the quantum raising operator is exactly the same as the classical raising operator, but it acts very quickly and there are some (unknown) physical effects or principles that cause the field to "click" into certain quantized ("particle") states.

The Connection: Graphing QFT Eigen-States

Another Possible Notation

Maybe the Schrödinger picture doesn't have these kind of states, but here's a notational possibility:

$$\begin{aligned}
\| \, \mathbf{\Phi}(t) \, \rangle\!\rangle &= \Big(\| \, \Phi_1(t) \, \rangle\!\rangle , \| \, \Phi_2(t) \, \rangle\!\rangle , \ldots , \| \, \Phi_n(t) \, \rangle\!\rangle , \ldots \Big) \\
\langle\!\langle \, \mathbf{x}, \phi \, \| \mathbf{\Phi}(t) \, \rangle\!\rangle &= \langle\!\langle \, \mathbf{x}, \phi \, \| \Big(\| \, \Phi_1(t) \, \rangle\!\rangle , \| \, \Phi_2(t) \, \rangle\!\rangle , \ldots , \| \, \Phi_n(t) \, \rangle\!\rangle , \ldots \Big) \\
&= \Big(\langle\!\langle \, \mathbf{x}, \phi \, \| \Phi_1(t) \, \rangle\!\rangle , \langle\!\langle \, \mathbf{x}, \phi \, \| \Phi_2(t) \, \rangle\!\rangle , \ldots , \langle\!\langle \, \mathbf{x}, \phi \, \| \Phi_n(t) \, \rangle\!\rangle , \ldots \Big) \\
&= \big(\Phi_1(\mathbf{x}, \phi, t), \Phi_2(\mathbf{x}, \phi, t), \ldots , \Phi_n(\mathbf{x}, \phi, t), \ldots \big) \quad = \quad \mathbf{\Phi}(\mathbf{x}, \phi, t) \\
\langle\!\langle \, \mathbf{p}, \phi \, \| \mathbf{\Phi}(t) \, \rangle\!\rangle &= \big(\Phi_1(\mathbf{p}, \phi, t), \Phi_2(\mathbf{p}, \phi, t), \ldots , \Phi_n(\mathbf{p}, \phi, t), \ldots \big) \quad = \quad \mathbf{\Phi}(\mathbf{p}, \phi, t) \\
\langle\!\langle \, \mathbf{x}, \pi \, \| \mathbf{\Phi}(t) \, \rangle\!\rangle &= \big(\Phi_1(\mathbf{x}, \pi, t), \Phi_2(\mathbf{x}, \pi, t), \ldots , \Phi_n(\mathbf{x}, \pi, t), \ldots \big) \quad = \quad \mathbf{\Phi}(\mathbf{x}, \pi, t) \\
\langle\!\langle \, \mathbf{p}, \pi \, \| \mathbf{\Phi}(t) \, \rangle\!\rangle &= \big(\Phi_1(\mathbf{p}, \pi, t), \Phi_2(\mathbf{p}, \pi, t), \ldots , \Phi_n(\mathbf{p}, \pi, t), \ldots \big) \quad = \quad \mathbf{\Phi}(\mathbf{p}, \pi, t) \\
\langle\!\langle \, \phi(\mathbf{x}) \, \| \mathbf{\Phi}(t) \, \rangle\!\rangle &= \big(\Phi_1(\phi(\mathbf{x}), t), \Phi_2(\phi(\mathbf{x}), t), \ldots , \Phi_n(\phi(\mathbf{x}), t), \ldots \big) \quad = \quad \mathbf{\Phi}(\phi(\mathbf{x}), t) \quad \neq \quad \mathbf{\Phi}(\mathbf{x}, \phi, t)
\end{aligned}$$

$$\int \mathcal{D}\phi \, \left| \mathbf{\Phi}[\phi](t) \right|^2 = 1 \qquad ?$$

$$\| \, \mathbf{0} \, \rangle\!\rangle \quad = \quad \big(\| \, 0_1 \, \rangle\!\rangle , \| \, 0_2 \, \rangle\!\rangle , \ldots , \| \, 0_n \, \rangle\!\rangle , \ldots \big)$$

$$\langle\!\langle \, 0_n \, \| 0_n \, \rangle\!\rangle = n?$$

Idea

Is it possible to look at QFT states as a "wave-function cloud" or meta-wave-function, with a complex number at each point in $(\mathbf{x}, \phi)$-space? If so, how does it relate to the functional description, where there is a complex number for each function? It seems that there must be many more functions than there are points in this space, so I'm not sure if a relation can exist. However, there are restrictions imposed on these functions, like, perhaps, they must be Fourier-transformable. Does this reduce the number of functions so that a relation can exist?

From Hatfield

See, e.g., equations (10.15), (10.17), and (10.28). The ground state wave-functional for the Klein-Gordon equation is

$$\begin{aligned}
\Phi_0[\phi] &= \eta \exp\big(-G[\phi]\big) \\
&= \eta \exp\left(-\int \mathrm{d}^3x\,\mathrm{d}^3y\ \phi(\mathbf{x})\,g(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\right) \qquad (1)\\
&= \eta \exp\left(-\int \mathrm{d}^3x\,\mathrm{d}^3y\ \phi(\mathbf{x})\left(\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{g}(\mathbf{k})\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\right)\phi(\mathbf{y})\right) \\
&= \eta \exp\left(-\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{g}(\mathbf{k})\left(\int \mathrm{d}^3x\ \phi(\mathbf{x})\,e^{-i(-\mathbf{k})\cdot\mathbf{x}}\right)\left(\int \mathrm{d}^3y\ \phi(\mathbf{y})\,e^{-i\mathbf{k}\cdot\mathbf{y}}\right)\right) \\
&= \eta \exp\left(-\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{g}(\mathbf{k})\,\tilde{\phi}(-\mathbf{k})\,\tilde{\phi}(\mathbf{k})\right) \\
\tilde{g}(\mathbf{k}) &= \frac{1}{2}\sqrt{\mathbf{k}^2+m^2} \;=\; \frac{1}{2}\omega_k \\
\eta &= \left(\prod_{\mathbf{k}}\left(\sqrt{\frac{\pi}{\omega_k}}\right)\right)^{-1/2} \;=\; \prod_{\mathbf{k}}\left(\frac{\omega_k}{\pi}\right)^{1/4}
\end{aligned}$$

$$\Phi_0[\tilde{\phi}] = \left(\prod_{\mathbf{k}}\left(\frac{\omega_k}{\pi}\right)^{1/4}\right)\ \exp\left(-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\omega_k\,\tilde{\phi}(\mathbf{k})\,\tilde{\phi}(-\mathbf{k})\right)$$

Taking Idea and Inspiration from Hatfield

The quantum field (or meta-wave-function):

$$\Phi = \Phi(\phi, x) = \Phi_x(\phi)$$

If the field is made of oscillators, one at each point x in space, then each of them has a wave-function $\Phi_x(\phi)$ over a non-spatial coordinate ϕ. The probability that the oscillator is at a position ϕ is proportional to $\Phi_x(\phi)$, so the probability that the field takes a certain configuration $\varphi(x)$ over space should be the product of all these point-wise probabilities. Thus the wave-functional is the "continuous product" of probability amplitudes[1] is

$$\Phi[\varphi] = \prod_x \Phi\big(x, \varphi(x)\big) = \prod_x e^{\ln \Phi\left(x,\varphi(x)\right)} = e^{\int \mathrm{d}x\ \ln \Phi\left(x,\varphi(x)\right)}.$$

The ground state for the field should be that each of these oscillators is in its ground state, so that

$$\Phi_x(\phi) = \Phi_{\mathrm{g}}(\phi) = \left(\frac{1}{(2\pi)^{1/4}\sqrt{\phi_0}}\right)\exp\left[-\frac{1}{4}\left(\frac{\phi}{\phi_0}\right)^2\right] = A\exp\left[-\frac{1}{4}\left(\frac{\phi}{\phi_0}\right)^2\right]$$

where perhaps $A = 1$. Thus we have

$$\Phi_0[\varphi] = e^{\int \mathrm{d}x\ \ln \Phi_{\mathrm{g}}(\varphi(x))} = e^{-\int \mathrm{d}x\ \ \varphi^2(x)/4\phi_0{}^2}$$

[1] Thanks go to Chris Clark for helping me get the final desired form.

which looks a lot like Equation 1. If we had $g(x,y) = \delta(x-y)/4{\phi_0}^2$ then they *would* be the same. It seems that $g(x,y)$ *not* being proportional to a delta function encapsulates some information about how the oscillators at different points interact and are coupled. But what is $g(x,y)$?

Let's take the 3-D example from Hatfield (10.21) and (10.22). Since $\tilde{g}(\mathbf{k}) = \frac{1}{2}\sqrt{\mathbf{k}^2 + m^2}$, we have

$$g(x,y) = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\tilde{g}(\mathbf{k})\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

But since $\tilde{g}(\mathbf{k})$ goes to infinity as k goes to infinity, this integral cannot converge, so g does not exist! What's wrong with this assessment?

Solve for Eigenstates using Raising/Lowering Operators

From Quantum Mechanics

We can use the annihilator a and creator $a^\dagger$ to find the energy eigenfunctions in the position representation (as well as the momentum representation). (Let $\mathrm{d}_x = \mathrm{d}/\mathrm{d}x$.)

$$\begin{aligned}\langle x|a|0\rangle &= \frac{1}{2}\left\langle x\left|\left(\frac{X}{x_0}+i\frac{P}{p_0}\right)\right|0\right\rangle = \frac{1}{2}\left(\frac{x}{x_0}+i(-i\hbar)\frac{\mathrm{d}_x}{p_0}\right)\langle x|0\rangle = \frac{1}{2}\left(\frac{x}{x_0}+(2x_0p_0)\frac{\mathrm{d}_x}{p_0}\right)\langle x|0\rangle \\ &= \langle x|\rangle = 0\end{aligned}$$

So

$$\left(x + 2{x_0}^2\mathrm{d}_x\right)\langle x|0\rangle = 0$$

and the normalized solution is

$$\langle x|0\rangle = \frac{1}{\pi^{1/4}\sqrt{\sqrt{2}\,x_0}}\exp\left[-\frac{1}{2}\left(\frac{x}{\sqrt{2}\,x_0}\right)^2\right] = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}}\exp\left[-\frac{1}{4}\left(\frac{x}{x_0}\right)^2\right]$$

so that the probability density

$$\left|\langle x|0\rangle\right|^2 = \frac{1}{\sqrt{2\pi}\,x_0}\exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

is a Gaussian distribution over space with a standard deviation of x_0.

Following that Pattern

$$\begin{aligned}
\left\langle \tilde{\phi} \middle| \hat{\pi}(\mathbf{p}) \middle| \tilde{\phi}' \right\rangle &= -i \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \; \delta[\tilde{\phi} - \tilde{\phi}'] \\
\hat{a}_{\mathbf{p}} &= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\hat{\varphi}(\mathbf{p}) + i \frac{\hat{\pi}(\mathbf{p})}{e_{\mathbf{p}}} \right) \\
\left\langle \tilde{\phi} \middle| \hat{a}_{\mathbf{p}} \middle| \tilde{\phi}' \right\rangle &= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \right) \delta[\tilde{\phi} - \tilde{\phi}'] \\
\left\langle \tilde{\phi} \middle| \hat{a}_{\mathbf{p}} \middle| \Phi_0 \right\rangle &= \int \mathcal{D}\tilde{\phi}' \; \left\langle \tilde{\phi} \middle| \hat{a}_{\mathbf{p}} \middle| \tilde{\phi}' \right\rangle \left\langle \tilde{\phi}' \middle| \Phi_0 \right\rangle \\
&= \int \mathcal{D}\tilde{\phi}' \; \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \right) \delta[\tilde{\phi} - \tilde{\phi}'] \; \Phi_0[\tilde{\phi}'] \\
&= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \right) \Phi_0[\tilde{\phi}] \\
&= 0
\end{aligned}$$

So, solve this:

$$\sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\tilde{\phi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \, \frac{\delta}{\delta \tilde{\phi}(\mathbf{p})} \right) \Phi_0[\tilde{\phi}] = 0$$

Bad notation

$$
\begin{aligned}
\mathbf{\Phi}[\phi](t) &= \vec{\Phi}[\phi](t) \\
&= \langle\!| \, \phi(t) \, \| \Phi(t) \, |\!\rangle \\
&= \Big(\big\langle \phi_1(t) \big| , \big\langle \phi_2(t) \big| , \dots , \big\langle \phi_n(t) \big| , \dots \Big) \Big(\big| \Phi_1(t) \big\rangle , \big| \Phi_2(t) \big\rangle , \dots , \big| \Phi_n(t) \big\rangle , \dots \Big) \\
&= \Big(\big\langle \phi_1(t) \big| \Phi_1(t) \big\rangle , \big\langle \phi_2(t) \big| \Phi_2(t) \big\rangle , \dots , \big\langle \phi_n(t) \big| \Phi_n(t) \big\rangle , \dots \Big) \\
&= \big(\Phi_1[\phi_1](t), \Phi_2[\phi_2](t), \dots , \Phi_n[\phi_n](t), \dots \big) \\
&\quad \Big(\text{This is unsatisfactory since the } \big| \Phi_n(t) \big\rangle \text{ are not normal Dirac states.} \Big) \\
&\quad \text{(Using capital versus lowercase letters seems} \\
&\quad \text{ to be the best notational option I can think of.)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_{\mathbf{p}} \, |\phi_{\mathbf{k}}\rangle &= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\hat{\varphi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \frac{\delta}{\delta \phi(\mathbf{p})} \right) |\phi_{\mathbf{k}}\rangle \\
&\quad \text{(Maybe there's no such thing as kets, } \hat{\mathbf{x}}, \text{ and } \hat{\mathbf{p}} \text{ in the Schrödinger functional picture.)} \\
\hat{a}_{\mathbf{p}} \, \Phi[\phi(\mathbf{k})] &= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\hat{\varphi}(\mathbf{p}) + \frac{1}{e_{\mathbf{p}}} \frac{\delta}{\delta \phi(\mathbf{p})} \right) \Phi[\phi(\mathbf{k})] \\
&= \sqrt{\frac{e_{\mathbf{p}}}{2}} \left(\phi(\mathbf{k}) + \frac{1}{e_{\mathbf{p}}} \frac{\delta}{\delta \phi(\mathbf{p})} \right) \Phi[\phi(\mathbf{k})]
\end{aligned}
$$

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)