Week 8 Lecture: Concepts of Quantum Field Theory (QFT)

Andrew Forrester February 29, 2008

Klein-Gordon Green's Functions and Raising/Lowering Operators

This Week's Questions

- How do the Green's functions of the classical Klein-Gordon field relate to the raising operators of the quantum Klein-Gordon field? (Maybe they don't really relate.)
  - Also, how does the propagator relate to the Green's function and raising/lowering operators?
- What are the general solutions to the Klein-Gordon equation?
  Clue: `http://www.jstor.org/view/00804630/di002669/00p0005t/0`
  A Klein-Gordon Model Particle, by J. L. Synge
  Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Copyright 1965 The Royal Society

Motivation / Inspiration

In Morse and Feshbach [1], pages 122-124, the connection between a concentrated force on a string and a Green's function is discussed. The force raises (or displaces) the string, of course, and this puts energy into the string system. This idea could work in the same way for field theory, so that external forces are what cause the creation of particles and "raise" the field.

Klein-Gordon Green's Function and Propagator

The Klein-Gordon operator $\mathrm{D}_{\text{K-G}}$ is

$$\mathrm{D}_{\text{K-G}} = \frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2$$

and the Klein-Gordon equation is

$$\mathrm{D}_{\text{K-G}} \left| f \right\rangle = 0.$$

However, in general we may have a source term $s(x)$, and this relates to the Green's functions $G(x, x')$ and $g(\mathbf{x}, \mathbf{x}')$:

$$\begin{aligned}
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] f(x) &= \frac{1}{\lambda}\, s(x) \qquad (1)\\
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] G(x, x') &= \delta(x - x')\\
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] g(\mathbf{x}, \mathbf{x}') = \left[-\boldsymbol{\nabla}^2 + \nu^2\right] g(\mathbf{x}, \mathbf{x}') &= \delta(\mathbf{x} - \mathbf{x}')
\end{aligned}$$

The general solution is given by

$$\begin{aligned}
f(x) &= \int \mathrm{d}^4x'\, \frac{1}{\lambda}\, s(x')\, G(x, x')\\
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] f(x) &= \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] \int \mathrm{d}^4x'\, \frac{1}{\lambda}\, s(x')\, G(x, x')\\
&= \int \mathrm{d}^4x'\, \frac{1}{\lambda}\, s(x') \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] G(x, x')\\
&= \int \mathrm{d}^4x'\, \frac{1}{\lambda}\, s(x')\, \delta(x - x')\\
&= \frac{1}{\lambda}\, s(x)
\end{aligned}$$

Time-independent, stationary solutions are given by

$$\begin{aligned}
f_{\text{stationary}}(\mathbf{x}) &= \int \mathrm{d}^3x'\, \frac{1}{\lambda}\, s_{\text{stationary}}(\mathbf{x}')\, g(\mathbf{x}, \mathbf{x}')\\
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] f_{\text{stationary}}(\mathbf{x}) &= \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] \int \mathrm{d}^3x'\, \frac{1}{\lambda}\, s_{\text{stationary}}(\mathbf{x}')\, g(\mathbf{x}, \mathbf{x}')\\
&= \int \mathrm{d}^3x'\, \frac{1}{\lambda}\, s_{\text{stationary}}(\mathbf{x}') \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] g(\mathbf{x}, \mathbf{x}')\\
&= \int \mathrm{d}^3x'\, \frac{1}{\lambda}\, s_{\text{stationary}}(\mathbf{x}')\, \delta(\mathbf{x} - \mathbf{x}')\\
&= \frac{1}{\lambda}\, s_{\text{stationary}}(\mathbf{x})
\end{aligned}$$

Let's try to solve explicitly for what $G(x, x')$ is for the Klein-Gordon equation.

If $G(x, x') = G(x - x')$,
$(x - x') = (t - t', \mathbf{x} - \mathbf{x}') = (\tau, \mathbf{R}) \to (R) = (v\tau, \mathbf{R})$,
$(q - q') = (\omega - \omega', \mathbf{q} - \mathbf{q}') = (\Omega, \mathbf{Q}) \to (Q) = (\Omega/v, \mathbf{Q})$,
$Q \cdot R = \Omega\tau - \mathbf{Q} \cdot \mathbf{R}$,

$R^2 = R \cdot R = v^2\tau^2 - \mathbf{R}^2$
$Q^2 = Q \cdot Q = \Omega^2/v^2 - \mathbf{Q}^2$

$$G(x, x') = G(R) = G(\tau, \mathbf{R})$$

then the Fourier transform of G is

$$\begin{aligned}
\tilde{G}(Q) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^4} \mathrm{d}^4R \; G(R)\, e^{iQ \cdot R} \\
G(R) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; \tilde{G}(Q)\, e^{-iQ \cdot R} \\
\delta(R) &= \frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; e^{-iQ \cdot R}
\end{aligned}$$

and noting this,

$$\left[\frac{1}{v^2}\partial_t{}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] e^{-iQ \cdot R} = \left[-\frac{\Omega^2}{v^2} + \mathbf{Q}^2 + \nu^2\right] e^{-iQ \cdot R} = \left[-Q^2 + \nu^2\right] e^{-iQ \cdot R}$$

we have

$$\begin{aligned}
\left[\frac{1}{v^2}\partial_t{}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] G(R) &= \delta(R) \\
\Rightarrow \quad \left[\frac{1}{v^2}\partial_t{}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] \left(\frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; \tilde{G}(Q)\, e^{-iQ \cdot R}\right) &= \left(\frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; e^{-iQ \cdot R}\right) \\
\Rightarrow \quad \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; \tilde{G}(Q) \left[\frac{1}{v^2}\partial_t{}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] e^{-iQ \cdot R} &= \frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; e^{-iQ \cdot R} \\
\Rightarrow \quad \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; \tilde{G}(Q) \left[-Q^2 + \nu^2\right] e^{-iQ \cdot R} &= \frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^4} \mathrm{d}^4Q \; e^{-iQ \cdot R} \\
\Rightarrow \quad \frac{1}{(2\pi)^2} \tilde{G}(Q) \left[-Q^2 + \nu^2\right] &= \frac{1}{(2\pi)^4} \\
\Rightarrow \quad \tilde{G}(Q) &= -\frac{1}{(2\pi)^2} \frac{1}{Q^2 - \nu^2}
\end{aligned}$$

This is the Fourier transform of the Klein-Gordon Green's function. It is a propagator that we see in quantum field theory.

Additional Notes and Scratch Work

pg ? Morse and Feshbach

$$
\begin{aligned}
G(x|\xi) &= \begin{cases} \frac{1}{2\nu} e^{\nu(x-\xi)}; & x < \xi \\ \frac{1}{2\nu} e^{\nu(\xi-x)}; & x > \xi \end{cases} \\
g(R,r) &= \frac{\delta[\tau - (R/c)]}{R} - \frac{\kappa}{\sqrt{\tau^2 - (R/c)^2}} J_1[\kappa c \sqrt{\tau^2 - (R/c)^2}] \cdot \Theta[\tau - (R/c)] \\
\tilde{G}(q) &= \int \mathrm{d}^4 q \; G(x) \, e^{iq \cdot x} \\
&= \frac{1}{q^2 - m^2} \\
\tilde{G}^{(+)}(q) &= \frac{1}{q^2 - m^2 + i\epsilon}
\end{aligned}
$$

pg 856 Morse and Feshbach (here, g means the Green's function over all space, with no boundary)
pg 370-371 Aitchison and Hey

$$
\begin{aligned}
(x) &= (t, \mathbf{x}) \\
&= (t, x^1, x^2, x^3) \\
(q) &= (\omega, \mathbf{q}) \\
&= (\omega, q^1, q^2, q^3) \\
\left[\frac{1}{v^2} {\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2 \right] f(x) &= \frac{1}{\lambda} \, s(x) \\
\left[\frac{1}{v^2} {\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2 \right] G(x, x') &= \delta(x - x') \\
\left[\frac{1}{v^2} {\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2 \right] g(\mathbf{x}, \mathbf{x}') &= \delta(\mathbf{x} - \mathbf{x}')
\end{aligned}
$$

[Should the measure be Lorentz invariant?]

$$
\begin{aligned}
\tilde{G}(q, q') &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^8} \mathrm{d}^4 x \, \mathrm{d}^4 x' \; G(x, x') \, e^{-iq \cdot x} \, e^{-iq' \cdot x'} \\
\tilde{G}^0(q, 0) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \mathrm{d}^4 x \; G(x, 0) \, e^{-iq \cdot x} \\
G(x, 0) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \mathrm{d}^4 q \; \tilde{G}^0(q, 0) \, e^{iq \cdot x} \\
\delta(x - x') &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \mathrm{d}^4 q \; e^{iq \cdot (x - x')}
\end{aligned}
$$

$$\begin{aligned}
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] G(x,0) &= \delta(x) \\
\Rightarrow \qquad \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] \left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{d}^4 q\ \tilde{G}(q,0)\, e^{iq\cdot x}\right) &= \left(\frac{1}{(2\pi)^4}\int_{\mathbb{R}^2} \mathrm{d}^4 q\ e^{iq\cdot x}\right) \\
\Rightarrow \qquad \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathrm{d}^4 q\ \tilde{G}(q,0)\left[-q^2+\nu^2\right] e^{iq\cdot x} &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2} \mathrm{d}^4 q\ e^{iq\cdot x} \\
\Rightarrow \qquad \frac{1}{(2\pi)^2}\,\tilde{G}(q,0)\left[-q^2+\nu^2\right] &= \frac{1}{(2\pi)^4} \\
\Rightarrow \qquad \tilde{G}(q,0) &= -\frac{1}{(2\pi)^2}\frac{1}{q^2-\nu^2}
\end{aligned}$$

[Should the measure be Lorentz invariant?]

$$\begin{aligned}
G(x-x') &= G(\mathbf{x}-\mathbf{x}', t-t') \\
&= G(\mathbf{R},\tau) \\
g(\mathbf{x}-\mathbf{x}') &= g(\mathbf{R}) \\
\hat{G}(\mathbf{R},\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}t\ G(\mathbf{R},\tau)\, e^{i\omega\tau} \\
G(\mathbf{R},\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \hat{G}(\mathbf{R},\omega)\, e^{-i\omega\tau} \\
\delta(\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-i\omega\tau}
\end{aligned}$$

$$\begin{aligned}
\left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right] G(x,x') &= \delta(x-x') \\
\Rightarrow \qquad \left[\frac{1}{v^2}{\partial_t}^2 - \boldsymbol{\nabla}^2 + \nu^2\right]\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \hat{G}(\mathbf{R},\omega)\, e^{-i\omega\tau}\right) &= \left(\delta(\mathbf{R})\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-i\omega\tau}\right) \\
\Rightarrow \qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \left[-\frac{\omega^2}{v^2} - \boldsymbol{\nabla}^2 + \nu^2\right]\hat{G}(\mathbf{R},\omega)\, e^{-i\omega\tau} &= \delta(\mathbf{R})\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-i\omega\tau} \\
\Rightarrow \qquad \left[-\boldsymbol{\nabla}^2 + (\nu^2 - \omega^2/v^2)\right]\hat{G}(\mathbf{R},\omega) &= \delta(\mathbf{R})
\end{aligned}$$

Green's Functions and the Real-Valued Classical Klein-Gordon Field

We add another assumption to our physical model for the real-valued classical Klein-Gordon field:

(9) external forces S (both attractive and repulsive) may be exerted on the sheet from above, but only in the z-direction

(the force area-density is $s = s(x, y, t)$. The origin of these forces is not contained in this model – let's just say someone's sticky fingers could be involved.)

The equation of motion is

$$F_z \quad = \quad (\mu\, \delta x\, \delta y)\, {\partial_t}^2 f,$$

with the forces given by

$$\begin{aligned}
F_z &= F_z^{\mathrm{s}} + F_z^{\mathrm{t}x} + F_z^{\mathrm{t}y} + S_z \\
F_z^{\mathrm{s}} &= -(\kappa\, \delta x\, \delta y) f \\
F_z^{\mathrm{t}x} &= (\lambda\, \delta y)\, \delta(\partial_x f) \\
F_z^{\mathrm{t}y} &= (\lambda\, \delta x)\, \delta(\partial_y f) \\
S_z &= \delta x\, \delta y\, s.
\end{aligned}$$

So we have

$$\begin{aligned}
F_z \quad = \quad -(\kappa\, \delta x\, \delta y) f + (\lambda\, \delta y)\, \delta(\partial_x f) + (\lambda\, \delta x)\, \delta(\partial_y f) + \delta x\, \delta y\, s \quad &= \quad (\mu\, \delta x\, \delta y)\, {\partial_t}^2 f \\
\Rightarrow \qquad -\frac{\kappa}{\lambda} f + \frac{\delta(\partial_x f)}{\delta x} + \frac{\delta(\partial_y f)}{\delta y} + \frac{1}{\lambda}\, s \quad &= \quad \frac{\mu}{\lambda}\, {\partial_t}^2 f,
\end{aligned}$$

after dividing by $\lambda\, \delta x\, \delta y$, and if we take the limit as $\delta x \to 0$ and $\delta y \to 0$, we get

$$-\frac{\kappa}{\lambda} f + {\partial_x}^2 f + {\partial_y}^2 f + \frac{1}{\lambda}\, s \quad = \quad \frac{\mu}{\lambda}\, {\partial_t}^2 f.$$

Thus, letting

$$v = \sqrt{\lambda/\mu} \qquad \nu = \sqrt{\kappa/\lambda}$$

$$\boxed{\frac{1}{v^2} {\partial_t}^2 f - \boldsymbol{\nabla}^2 f + \nu^2 f \quad = \quad \frac{1}{\lambda}\, s}$$

we have an inhomogeneous Klein-Gordon-type equation with a "source" function s/λ.

Let's compare these equations:

$$\begin{aligned}
\frac{1}{v^2} {\partial_t}^2 f - \boldsymbol{\nabla}^2 f &= 0 && \text{(Wave Equation; sheet without springy slab)} \\
\frac{1}{v^2} {\partial_t}^2 f - \boldsymbol{\nabla}^2 f + \nu^2 f &= 0 && \text{(Klein-Gordon-type equation; sheet with springy slab)} \\
\frac{1}{c^2} {\partial_t}^2 \varphi - \boldsymbol{\nabla}^2 \varphi + \nu^2 \varphi &= 0 && \text{(Klein-Gordon Equation)}
\end{aligned}$$

where $\nu = mc/\hbar$ for the Klein-Gordon equation, since it is describing the field φ of a particle of mass m.

$$\begin{aligned}
\frac{1}{c^2} {\partial_t}^2 \varphi \ &: \ \text{the generalized momentum-density rate of change} \\
\boldsymbol{\nabla}^2 \varphi \ &: \ \text{the generalized spring-force-density (perpendicular to spacetime)} \\
\nu^2 \varphi \ &: \ \text{the generalized surface-tension-force-density (perpendicular to spacetime)}
\end{aligned}$$

References

[1] Morse, Feshbach: *Methods of Theoretical Physics, Part 1*, McGraw-Hill Book Company, Inc. (1953)

- This book is very good. It takes a physically-based (as opposed to purely mathematical) approach to understanding the mathematics of physics and helps to create intuition.

[2] I. J. R. Aitchison, A. J. G. Hey: *Gauge Theories in Particle Physics, A Practical Introduction, Third Edition. Volume I: From Relativistic Quantum Mechanics to QED*, Taylor & Francis Group, LLC (2003)

- Appendix G is a good, quick description of a few Green's function examples, including that for the Klein-Gordon equation.
- Appendix F, on contour integration, may be helpful in understanding the complex versions of the Green's functions / propagators.

[3] Economou, L. N.: *Green's Functions in Quantum Physics, Second Corrected and Updated Edition*, (Springer Series in Solid-State Sciences 7), Springer-Verlag (1983)

- The first chapter seems to give a good explanation of the general formalism for Green's functions, although I didn't have time to go through it carefully.
- The titles of each part are:
  - Part I: Green's Functions in Mathematical Physics
  - Part II: Green's Functions in One-Body Quantum Problems
  - Part III: Green's Functions in Many-Body Systems