Week 7 Lecture: Concepts of Quantum Field Theory (QFT)

Andrew Forrester February 22, 2008

Deriving the Klein-Gordon Equation with a Physical Model

This Week's Questions/Goals

- How do you derive the Klein-Gordon equation (or a Klein-Gordon-type equation) using a physical model?
- Follow procedure to get creation/annihilation operators for the other equations.
  - I think after you've gotten it for the complex-valued Klein-Gordon quantum field, you've got it for all cases. (I will have to check that the result agrees with the Schrödinger mechanics case.)
- What is the position-eigenstate creation operator for any mechanics? Is it $\phi^*(x,t)$ in all cases? What about mixing momentum-eigenstate creation and annihilation operators?
- ⋆ Problem: I'm having trouble deriving the Schrödinger and Klein-Gordon equations from their Lagrangian (densities).

A Physical Model for the Real-Valued Classical Klein-Gordon Field

To understand the Klein-Gordon equation of quantum field theory, we should understand its classical incarnation. To approach an understanding of the classical Klein-Gordon equation, in particular its $(2+1)$-D form, we will examine a seemingly unlikely model, which is essentially continuously coupled harmonic oscillators. This will give us some intuition for the behavior of fields, and what assumptions those behaviors depend upon.

We examine the forces on a 2-D elastic sheet (thin but infinite in extent) that is glued to an elastic, solid slab (of finite thickness but infinite in extent), which is glued to a firm, solid base (infinite in extent). The slab is essentially[1] a continuous array of springs with spring-parameter area-density κ, meaning a chunk of cross-section $\delta x \times \delta y$ has a spring constant of $\delta k = \kappa\, \delta x\, \delta y$. We make the following important assumptions (which are not all independent):

(1) the elastic slab (and glue) is very light (of negligible mass compared to the sheet)

(2) the springy-ness of the slab is homogeneous

(the slab's spring-parameter area-density κ is constant)

(3) the deviations of the sheet are never too far from the equilibrium position

(so Hooke's law never breaks down anywhere in the slab)

(4) the deviations of nearby points of the sheet are not extremely different

(we will make small-angle, or small-slope arguments)

(5) the mass distribution of the elastic sheet is homogeneous

(the sheet's mass area-density μ is constant)

(6) the tension in the sheet is essentially constant everywhere

(the sheet's tension length-density λ is constant)

(7) the sheet moves mostly perpendicularly to the firm base (and barely moves side-to-side)

(the sheet moves in the z-direction, but not the x- or y-directions; the deviations $f = f(x, y, t)$ of the sheet are thus real numbers, rather than vectors, with equilibrium position being $f = 0$)

(8) there is little friction or radiation

(practically no energy is lost by friction or radiative processes as heat)

The equation of motion of this system given by Newton's second law. Since essentially all of the mass is localized in the sheet, we need only examine the coordinates giving the location of each piece of the sheet. (It's a simple matter to describe the expansion and compression of the massless slab once the sheet's configuration is known.) Since the motions are only in the z-direction, the net forces on each piece of the sheet are given by their z-components. So the net force on a small $\delta x \times \delta y$ piece of the sheet is

$$F_z \;=\; \delta m\, \mathrm{d}_t{}^2 f \;\approx\; \delta m\, \partial_t{}^2 f \;=\; (\mu\, \delta x\, \delta y)\, \partial_t{}^2 f,$$

where $\mathrm{d}_t \equiv \mathrm{d}/\mathrm{d}t \approx \partial_t \equiv \partial/\partial t$ because the x- and y-positions of each piece of the sheet are essentially constant and independent of time:

$$\mathrm{d}_t f = \mathrm{d}_x f\, \cancelto{0}{\mathrm{d}_t x} + \mathrm{d}_y f\, \cancelto{0}{\mathrm{d}_t y} + \partial_t f.$$

The individual forces on this small piece of the sheet are the spring force from the slab immediately below, F_z^{s}, the net tension-force from the sheet pulling on the sides of constant x-value, $F_z^{\mathrm{t}x}$, and the net

[1] I say "essentially" because, with an array of springs, you may easily enforce that the springs only oscillate up-and-down by construction, but with the slab, one can see how this assumption could break down as the dynamics become more violent.

tension-force from the sheet pulling on the sides of constant y-value, $F_z^{\mathrm{t}y}$:

$$\begin{aligned}
F_z &= F_z^{\mathrm{s}} + F_z^{\mathrm{t}x} + F_z^{\mathrm{t}y} \\
F_z^{\mathrm{s}} &= -\delta k\, f \;=\; -(\kappa\,\delta x\,\delta y) f \\
F_z^{\mathrm{t}x} &= T\sin\theta_1^x - T\sin\theta_2^x \;=\; T\,\delta(\sin\theta^x) \;\approx\; T\,\delta(\tan\theta^x) \;=\; T\,\delta(\partial_x f) \;=\; (\lambda\,\delta y)\,\delta(\partial_x f) \\
F_z^{\mathrm{t}y} &= T\sin\theta_1^y - T\sin\theta_2^y \;=\; T\,\delta(\sin\theta^y) \;\approx\; T\,\delta(\tan\theta^y) \;=\; T\,\delta(\partial_y f) \;=\; (\lambda\,\delta x)\,\delta(\partial_y f)
\end{aligned}$$

We used our small-deviation (and thus small-angle) assumption (4) to write $\sin\theta \approx \tan\theta$. So Newton's second law gives us

$$\begin{aligned}
F_z \;=\; -(\kappa\,\delta x\,\delta y) f + (\lambda\,\delta y)\,\delta(\partial_x f) + (\lambda\,\delta x)\,\delta(\partial_y f) &= (\mu\,\delta x\,\delta y)\,\partial_t{}^2 f \\
\Rightarrow \qquad -\frac{\kappa}{\lambda} f + \frac{\delta(\partial_x f)}{\delta x} + \frac{\delta(\partial_y f)}{\delta y} &= \frac{\mu}{\lambda}\,\partial_t{}^2 f,
\end{aligned}$$

after dividing by $\lambda\,\delta x\,\delta y$, and if we take the limit as $\delta x \to 0$ and $\delta y \to 0$, we get

$$-\frac{\kappa}{\lambda} f + \partial_x{}^2 f + \partial_y{}^2 f \;=\; \frac{\mu}{\lambda}\,\partial_t{}^2 f.$$

Thus, letting

$$v = \sqrt{\lambda/\mu} \qquad \nu = \sqrt{\kappa/\lambda}$$

$$\boxed{\frac{1}{v^2}\partial_t{}^2 f - \boldsymbol{\nabla}^2 f + \nu^2 f \;=\; 0}$$

we have a Klein-Gordon-type equation.

Let's compare this equation to two others:

$$\begin{aligned}
\frac{1}{v^2}\partial_t{}^2 f - \boldsymbol{\nabla}^2 f &= 0 && \text{(Wave Equation; sheet without springy slab)} \\
\frac{1}{v^2}\partial_t{}^2 f - \boldsymbol{\nabla}^2 f + \nu^2 f &= 0 && \text{(Klein-Gordon-type equation; sheet with springy slab)} \\
\frac{1}{c^2}\partial_t{}^2 \varphi - \boldsymbol{\nabla}^2 \varphi + \nu^2 \varphi &= 0 && \text{(Klein-Gordon Equation)}
\end{aligned}$$

where $\nu = mc/\hbar$ for the Klein-Gordon equation, since it is describing the field φ of a particle of mass m.

The velocity v in our derived equation is the speed at which waves would travel in the sheet if it were not glued to the elastic slab. (The sheet would be just as taut without the slab as with it, and note that the sheet wouldn't be sagging because gravity is not in this picture at all.) Since there are forces in addition to the tension pulling the sheet toward equilibrium, this should have an effect similar to increasing the tension, so that the (phase) velocity of wave propagation is greater than v. But it's not the same as increasing the tension because, unlike the wave equation, our derived Klein-Gordon-type equation is dispersive, meaning waves of different frequencies travel at different speeds. As I show below (in the "Phase and Group Velocities" section), ν and v together define a critical frequency, above which signals may propagate but below which they decay. [2]

We now see that we can think of the (2+1)-D classical Klein-Gordon equation as the equation of motion for a 2-D elastic sheet (thin but infinite in extent) that is glued to an elastic, solid slab (of finite thickness but infinite in extent), which is glued to a firm, solid base (infinite in extent).[3] [4] Furthermore, we can give

[2] My initial thought was this: The spatial frequency ν is perhaps related to the uncertainty in the sheet-waves. (Perhaps ν gives the extent or standard deviation of a Gaussian distribution?)

[3] Actually, we do not have to have a firm base in this picture – the sheet could be imbedded in an infinite elastic medium, both above and below the sheet. It turns out, though, that the picture we've used here is convenient for an additional concept we add in the next lecture.

[4] If we rip off the rubbery sheet, will we be left with the Higgs field, sitting on top of the firm vacuum?

what might be a physical interpretation for each term:

$$\frac{1}{c^2}\partial_t{}^2\varphi \quad : \quad \text{the generalized momentum-density-per-tension rate of change}$$
$$\boldsymbol{\nabla}^2\varphi \quad : \quad \text{the generalized spring-force-density-per-tension (perpendicular to spacetime)}$$
$$\nu^2\varphi \quad : \quad \text{the generalized surface-tension-force-density-per-tension (perpendicular to spacetime)}$$

We've simply taken what was a momentum or force in our model to be an appropriately generalized momentum or force in the Klein-Gordon case. For our model, "spacetime" was the x-y-t-space, and the z-direction was perpendicular to spacetime. In the Klein-Gordon case, whatever dimension the field is oscillating in is taken to be that perpendicular direction.

After we've moved on to the quantum Klein-Gordon field (and beyond) and have a good understanding of quantum field theory, it may be wise to come back to the assumptions we made when deriving the Klein-Gordon equation and re-assess their validity for quantum fields.

Phase and Group Velocities

Although we are examining real-valued functions f, we can use complex functions $\underset{\sim}{f}$ to simplify the analysis of planewaves. Then we can analyze the behavior of more complicated functions using Fourier analysis.

$$\begin{aligned}\underset{\sim}{f} &= \underset{\sim}{f}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\\ f &= \operatorname{Re} \underset{\sim}{f}\end{aligned}$$

$$\begin{aligned}\frac{1}{v^2}\partial_t{}^2\underset{\sim}{f} - \boldsymbol{\nabla}^2\underset{\sim}{f} + \nu^2\underset{\sim}{f} &= 0\\ &= -\frac{1}{v^2}\omega^2\underset{\sim}{f} + \mathbf{k}^2\underset{\sim}{f} + \nu^2\underset{\sim}{f}\\ \Rightarrow \qquad -\frac{\omega^2}{v^2} + \mathbf{k}^2 + \nu^2 &= 0\end{aligned}$$

$$\begin{aligned}\omega &= \pm v\sqrt{\mathbf{k}^2+\nu^2}\\ \mathbf{k} &= k\hat{\mathbf{k}} \;=\; \sqrt{(\omega/v)^2-\nu^2}\,\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}v_\mathrm{p} &= \frac{\omega}{k}\\ &= \pm v\,\frac{\sqrt{\mathbf{k}^2+\nu^2}}{k} \;=\; \pm v\,\sqrt{1+(\nu/k)^2} \;=\; \pm v\left[1+\left(\nu\frac{1}{k}\right)^2\right]^{1/2}\\ &= \frac{\omega}{\sqrt{(\omega/v)^2-\nu^2}} \;=\; \pm v\,\frac{1}{\sqrt{1-(\nu v/\omega)^2}} \;=\; \pm v\left[1-\left(\nu\frac{v}{\omega}\right)^2\right]^{-1/2}\\ v_\mathrm{g} &= \frac{\mathrm{d}\omega}{\mathrm{d}k} \;=\; \frac{\mathrm{d}}{\mathrm{d}k}\left[\pm v\sqrt{\mathbf{k}^2+\nu^2}\right] \;=\; \pm v\frac{k}{\sqrt{\mathbf{k}^2+\nu^2}}\\ &= \pm v\left[1+\left(\nu\frac{1}{k}\right)^2\right]^{-1/2} \;=\; \pm v\left[1-\left(\nu\frac{v}{\omega}\right)^2\right]^{1/2}\end{aligned}$$

So we can see that there is a critical frequency $\omega_\mathrm{c} = \nu v$ that is not allowed to propagate since $k = 0$ (infinite wavelength), $v_\mathrm{g} = 0$, and $v_\mathrm{p} = \infty$. Any frequencies below this critical frequency have an imaginary wave-vector $\underset{\sim}{\mathbf{k}}$, and so they decay. But above this frequency, disturbances in the sheet or "signals" propagate freely. Tying this back to the Klein-Gordon equation, we see that signals with energy $\hbar\omega$ less than the rest energy mc^2 decay, signals with exactly the rest energy do not propagate, and signals with energy greater than the rest energy propagate freely (at a particular phase velocity). A more complicated signal with frequencies higher and lower than the critical frequency will have the lower frequencies eliminated while the higher frequencies disperse.

One interesting question is, are these decaying signals virtual particles? Well, the imaginary $\underset{\sim}{k}$ for these signals ranges from 0 to $i\nu$, so that means that the attenuation parameter $\alpha = i/\underset{\sim}{k}$ ranges from ∞ to $1/\nu$. That suggests that this "virtual particle" of a field of mass-parameter m can reach out to infinity. But we know that only the virtual particles for fields of mass-parameter 0 can reach out to infinity. So it seems that these decaying signals do not correspond to virtual particles.

Large Deviations (Dropping Assumtion 4)

If we assume that the deviations can be large (rejecting assumption 4), but still retain the other assumptions, including that the deviations are still only in the z-direction (assumption 6), then we'll get a very different equation of motion. The sheet still can never bend back over itself, but now the slope of the sheet in any direction can be anywhere in the range $(-\infty, \infty)$ and the angles associated with these slopes are in the range $(-\pi/2, \pi/2)$. [5]

$$
\begin{aligned}
\sin\theta &= o/h \\
\cos\theta &= a/h \\
\tan\theta &= o/a \\
a &> 0 \\
\text{sgn } a &= 1 \\
h &= \sqrt{o^2 + a^2} \quad > \quad 0 \\
\sin\theta &= \frac{o}{h} = \frac{o}{\sqrt{o^2+a^2}} = \frac{o/a}{(\text{sgn } a)\sqrt{(o/a)^2+1}} = \frac{o/a}{\sqrt{1+(o/a)^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}
\end{aligned}
$$

$$
\begin{aligned}
F_z^{\text{t}x} &= T\sin\theta_1^x - T\sin\theta_2^x = T\,\delta(\sin\theta^x) = T\,\delta\left(\frac{\tan\theta^x}{\sqrt{1+\tan^2\theta^x}}\right) = T\,\delta\left(\frac{\partial_x f}{\sqrt{1+(\partial_x f)^2}}\right) \\
&= (\lambda\,\delta y)\,\delta\left(\frac{\partial_x f}{\sqrt{1+(\partial_x f)^2}}\right) \\
F_z^{\text{t}y} &= (\lambda\,\delta x)\,\delta\left(\frac{\partial_y f}{\sqrt{1+(\partial_y f)^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_z &= -(\kappa\,\delta x\,\delta y) f + (\lambda\,\delta y)\,\delta\left(\frac{\partial_x f}{\sqrt{1+(\partial_x f)^2}}\right) + (\lambda\,\delta x)\,\delta\left(\frac{\partial_y f}{\sqrt{1+(\partial_y f)^2}}\right) = (\mu\,\delta x\,\delta y)\,{\partial_t}^2 f \\
\Rightarrow \quad -\frac{\kappa}{\lambda} f &+ \frac{\delta\left(\frac{\partial_x f}{\sqrt{1+(\partial_x f)^2}}\right)}{\delta x} + \frac{\delta\left(\frac{\partial_y f}{\sqrt{1+(\partial_y f)^2}}\right)}{\delta y} = \frac{\mu}{\lambda}\,{\partial_t}^2 f,
\end{aligned}
$$

[5]For this to remain a realistic physical model, the deviations should not be too extreme, or else the slab would rip apart or the Hooke's law relation for the restoring force from the slab would become invalid.

$$
\begin{aligned}
\partial_x\left(\frac{\partial_x f}{\sqrt{1+(\partial_x f)^2}}\right) &= \frac{{\partial_x}^2 f}{\sqrt{1+(\partial_x f)^2}} + \partial_x f\left(-\frac{1}{2}\right)\frac{2(\partial_x f)\left({\partial_x}^2 f\right)}{\left[1+(\partial_x f)^2\right]^{3/2}} \\
&= \left({\partial_x}^2 f\right)\left[\frac{1}{\sqrt{1+(\partial_x f)^2}} - \frac{(\partial_x f)^2}{\left[1+(\partial_x f)^2\right]^{3/2}}\right] \\
&= \left({\partial_x}^2 f\right)\left[\frac{1+(\partial_x f)^2}{\left[1+(\partial_x f)^2\right]^{3/2}} - \frac{(\partial_x f)^2}{\left[1+(\partial_x f)^2\right]^{3/2}}\right] \\
&= \frac{{\partial_x}^2 f}{\left[1+(\partial_x f)^2\right]^{3/2}}
\end{aligned}
$$

$$
\Rightarrow \qquad -\frac{\kappa}{\lambda} f + \frac{{\partial_x}^2 f}{\left[1+(\partial_x f)^2\right]^{3/2}} + \frac{{\partial_y}^2 f}{\left[1+(\partial_y f)^2\right]^{3/2}} = \frac{\mu}{\lambda}\,{\partial_t}^2 f
$$

$$
\boxed{\frac{1}{v^2}\,{\partial_t}^2 f - \frac{{\partial_x}^2 f}{\left[1+(\partial_x f)^2\right]^{3/2}} - \frac{{\partial_y}^2 f}{\left[1+(\partial_y f)^2\right]^{3/2}} + \nu^2 f = 0}
$$

Let's get an expansion for relatively small deviations:

$$
\frac{{\partial_x}^2 f}{\left[1+(\partial_x f)^2\right]^{3/2}} = {\partial_x}^2 f\left[1 - \frac{3}{2}(\partial_x f)^2 + \frac{15}{8}(\partial_x f)^4 - \cdots\right]
$$

$$
\boxed{\frac{1}{v^2}\,{\partial_t}^2 f - \boldsymbol{\nabla}^2 f + {\partial_x}^2 f\left[\frac{3}{2}(\partial_x f)^2 - \frac{15}{8}(\partial_x f)^4 + \cdots\right] + {\partial_y}^2 f\left[\frac{3}{2}(\partial_y f)^2 - \frac{15}{8}(\partial_y f)^4 + \cdots\right] + \nu^2 f = 0}
$$

The Complex-Valued Schrödinger Quantum Field in the Schrödinger Picture

$$\left(\hat{E} - \left[\frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}\right]\right)\hat{\varphi} \quad = \quad \hat{0}$$

$$\left(i\hbar\,\partial_t - \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\mathbf{x})\right]\right)\hat{\varphi}(\mathbf{x}) \quad = \quad \hat{0}$$

$$\left(i\hbar\,\partial_t - \left[\frac{\mathbf{p}^2}{2m} + \tilde{V}(\mathbf{p})\right]\right)\hat{\varphi}(\mathbf{p}) \quad = \quad \hat{0}$$

where $\hat{E}$ is the energy operator and $\hat{\mathbf{p}}$ is the non-relativistic momentum operator.

$$\begin{aligned}
\hat{0} &= \left(\hat{E} - \frac{\hat{\mathbf{p}}^2}{2m} - \hat{V}\right)\hat{\varphi} \\
\hat{\mathcal{L}}_{\text{Schrö}} &= \\
\hat{\mathcal{H}}_{\text{Schrö}} &=
\end{aligned}$$

$$\begin{aligned}
\hat{0} &= \left(i\hbar\,\partial_t + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 - V(\mathbf{x})\right)\hat{\varphi}(\mathbf{x}) \\
\hat{\mathcal{L}}_{\text{Schrö}} &= \frac{i\hbar}{2}\left(\hat{\varphi}^*\partial_t\hat{\varphi} - \hat{\varphi}\,\partial_t\hat{\varphi}^*\right) - \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\hat{\varphi}^*)(\boldsymbol{\nabla}\hat{\varphi}) + V(x)\,\hat{\varphi}^*\hat{\varphi} \\
\hat{\mathcal{H}}_{\text{Schrö}} &= \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\hat{\varphi}^*)(\boldsymbol{\nabla}\hat{\varphi}) + V(x)\,\hat{\varphi}^*\hat{\varphi}
\end{aligned}$$

From Burgess [3] (pg 410) and Hatfield [1] (pg 23)

$$\begin{aligned}
S &= \int \mathrm{d}\sigma\,\mathrm{d}t\left\{\frac{i\hbar}{2}\left(\varphi^*\tilde{\partial}_t\varphi - \varphi\,\tilde{\partial}_t\varphi^*\right) - \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\varphi)^*(\boldsymbol{\nabla}\varphi) - V\,\varphi^*\varphi - J^*\varphi - \varphi^* J\right\} \\
\mathcal{L} &= \frac{i}{2}\left(\varphi^*\partial_t\varphi - \varphi\,\partial_t\varphi^*\right) - \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) + V(x)\,\varphi^*\varphi
\end{aligned}$$

(What's with this disagreement?)

The Real-Valued Klein-Gordon Quantum Field in the Schrödinger Picture

Analysis

(1) Start with equation of motion, Lagrangian, Hamiltonian

$$\left(\hat{E}^2 - \left[\hat{\mathbf{p}}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi} \;=\; \left(\hat{p}^2 c^2 - m^2 c^4\right)\hat{\varphi} \;=\; \hat{0}$$

$$-\hbar^2 c^2 \left(\partial^2 + \nu^2\right)\hat{\varphi}(\mathbf{x}) \;=\; \hat{0}$$

$$-\left(\hbar^2 {\partial_t}^2 + \left[\mathbf{p}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi}(\mathbf{p}) \;=\; \hat{0}$$

where $\hat{E}$ is the energy operator and $\hat{\mathbf{p}}$ and $\hat{p}_\mu$ are the relativistic momenta operators, $\hat{p}^2 = \hat{p}_\mu \hat{p}^\mu = (i\hbar\partial)^2 = -\hbar^2\partial^2$, $\partial^2 = \Box = \frac{1}{c^2}{\partial_t}^2 - \nabla^2$, and $\nu \equiv mc/\hbar$ is the Compton spatial frequency for the particle of mass m.

$$\begin{aligned}
\hat{0} &= \left(\hat{p}^2 - m^2\right)\hat{\varphi} \\
\hat{\mathcal{L}}_{\text{K-G}} &= \frac{1}{2}\left(\hat{p}^2 - m^2\right)\hat{\varphi}^2 \\
&= \frac{1}{2}\left(\hat{E}^2 - \hat{\mathbf{p}}^2 - m^2\right)\hat{\varphi}^2 \\
\hat{\mathcal{H}}_{\text{K-G}} &= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\hat{\mathbf{p}}^2 + m^2\right)\hat{\varphi}^2
\end{aligned}$$

$$\begin{aligned}
\hat{0} &= \left(\partial^2 + m^2\right)\hat{\varphi}(\mathbf{x}) \\
\hat{\mathcal{L}}_{\text{K-G}} &= \frac{1}{2}(\partial\hat{\varphi})^2 - \frac{1}{2}(m\hat{\varphi})^2 \\
&= \frac{1}{2}(\partial_t\hat{\varphi})^2 - \frac{1}{2}(\boldsymbol{\nabla}\hat{\varphi})^2 - \frac{1}{2}(m\hat{\varphi})^2 \qquad \text{not } -\frac{1}{2}(\partial_t\hat{\varphi})^2 \text{ ?} \\
\hat{\mathcal{H}}_{\text{K-G}} &= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\hat{\varphi})^2 + \frac{1}{2}(m\hat{\varphi})^2
\end{aligned}$$

$$\begin{aligned}
\hat{0} &= \left(p^2 - m^2\right)\hat{\varphi}(\mathbf{p}) \\
\hat{\mathcal{L}}_{\text{K-G}} &= \frac{1}{2}(p\hat{\varphi})^2 - \frac{1}{2}(m\hat{\varphi})^2 \\
&= \frac{1}{2}(\partial_t\hat{\varphi})^2 - \frac{1}{2}(\mathbf{p}\hat{\varphi})^2 - \frac{1}{2}(m\hat{\varphi})^2 \\
\hat{\mathcal{H}}_{\text{K-G}} &= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\mathbf{p}\hat{\varphi})^2 + \frac{1}{2}(m\hat{\varphi})^2
\end{aligned}$$

Stress-Energy-Momentum Tensor and Four Conserved Noether Charges:

$$\begin{aligned}
{T^\mu}_\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \mathcal{L}\,{\delta^\mu}_\nu \\
H &= \int \mathrm{d}^3x\; T^{00} \;=\; \int \mathrm{d}^3x\; \mathcal{H} \\
P^i &= \int \mathrm{d}^3x\; T^{0i} \;=\; \int \mathrm{d}^3x\; \pi\,\partial_i\phi
\end{aligned}$$

"The conserved charge H associated with time translations is the Hamiltonian (with current being power?). The conserved charges P^i associated with spatial translations are naturally interpreted as the (physical) momentum carried by the field (with current being force?), not to be confused with the canonical momentum (π?)." [6]

(2) Keep Lorentz invariance and relativistic causality in mind

(3) Derive creation and annihilation operators (make initial interpretation)

(4) Enforce causality on measurements made at spacelike-distant space-time points

(5) Interpret creation and annihilation operators

(6) Interpret field operator actions

(7) Determine conserved quantities and what they represent

(interpretations may have to wait until a gauge field is added and we compare with the complex-valued Klein-Gordon quantum field)

[6] Peskin & Schroeder, pg 19

Potential "Fock braket" Notation

Getting functions that you can graph (and animate over time) from a general QFT state:

$$\begin{aligned}
\| \Psi(t) \rangle\!\rangle &= \big(|\Psi_1(t)\rangle , |\Psi_2(t)\rangle , \ldots , |\Psi_n(t)\rangle , \ldots \big) \\
\langle\!\langle \mathbf{x} \| &= (\langle \mathbf{x}_1 | , \langle \mathbf{x}_1, \mathbf{x}_2 | , \ldots , \langle \mathbf{x}_1, \mathbf{x}_2, \ldots , \mathbf{x}_n | , \ldots) \\
\langle\!\langle \mathbf{x} \| \Psi(t) \rangle\!\rangle &= \big(\langle \mathbf{x}_1 | \Psi_1(t) \rangle , \langle \mathbf{x}_1, \mathbf{x}_2 | \Psi_2(t) \rangle , \ldots , \langle \mathbf{x}_1, \mathbf{x}_2, \ldots , \mathbf{x}_n | \Psi_n(t) \rangle , \ldots \big) \\
&= (\Psi_1(\mathbf{x}_1, t), \Psi_2(\mathbf{x}_1, \mathbf{x}_2, t), \ldots , \Psi_n(\mathbf{x}_1, \mathbf{x}_2, \ldots , \mathbf{x}_n, t), \ldots)
\end{aligned}$$

$$\begin{aligned}
|\Psi_n\rangle &= \int \mathrm{d}^3x_1\, \mathrm{d}^3x_2 \cdots \mathrm{d}^3x_n \;\; \Psi_n(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, t) \;\; \varphi^\dagger(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \;\; |0\rangle \\
&= \int \mathrm{d}^3p_1\, \mathrm{d}^3p_2 \cdots \mathrm{d}^3p_n \;\; \tilde{\Psi}_n(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n, t) \;\; a^\dagger(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) \;\; |0\rangle
\end{aligned}$$

$$\begin{aligned}
\varphi^\dagger(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) &= \varphi^\dagger(\mathbf{x}_1)\, \varphi^\dagger(\mathbf{x}_2) \cdots \varphi^\dagger(\mathbf{x}_n) \\
a^\dagger(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) &= a^\dagger(\mathbf{p}_1)\, a^\dagger(\mathbf{p}_2) \cdots a^\dagger(\mathbf{p}_n)
\end{aligned}$$

$$\begin{aligned}
\|\Psi\rangle\!\rangle &= |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\
&= \big(|\Psi_1\rangle , |\Psi_2\rangle , \ldots , |\Psi_n\rangle , \ldots \big) \\
&= \begin{pmatrix} 1 & 2 & 7 & n \\ f & g & r & s \end{pmatrix} = \begin{pmatrix} \text{number of particles } n \\ n\text{-particle wave-function} \end{pmatrix} \qquad \text{(ignoring all zero-functions)} \\
&= \|\Psi_1\rangle\!\rangle + \|\Psi_2\rangle\!\rangle + \cdots + \|\Psi_n\rangle\!\rangle + \cdots \\
\|\Psi_n\rangle\!\rangle &= |0\rangle \oplus |0\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\
&= \begin{pmatrix} n \\ f \end{pmatrix} \\
\|\Psi\rangle\!\rangle &= \begin{pmatrix} 1 & 2 & 7 & n \\ 2 & 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} \text{energy eigenstate index } n \\ \text{number of particles } m_n \end{pmatrix} \qquad \text{(ignoring all unoccupied eigenstates)} \\
&= (2, 2, 0, 0, 0, 0, 1, 0, 0, \ldots , 0, 5, 0, 0, \ldots) \\
&= (m_1, m_2, \ldots , m_n, \ldots) \\
\langle\!\langle \Psi \| \Psi \rangle\!\rangle &\stackrel{?}{=} \sum_n m_n
\end{aligned}$$

One could call the state above an "integer-number-eigen-particle" state.

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

[2] Michael E. Peskin, Daniel V. Schroeder: *An Introduction to Quantum Field Theory*, Westview Press (1995)

[3] Mark Burgess: *Classical Covariant Fields*, Cambridge University Press (2002)

[4] Resnick, Haliday, Krane: *Physics, 4th Edition, Volume 1*, John Wiley & Sons, Inc. (1992)

- Section 19-5 (page 425) "The Wave Equation" has a physically based derivation of the wave equation using a string. This inspired my physically based derivation of the Klein-Gordon equation.