Week 6 Lecture: Concepts of Quantum Field Theory (QFT)

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Progress Report on Work in QFT

Reading

This week I did more reading than usual and have summarized some of it in the references at the end of the paper. The following are essentially notes of the work that I did do.

This Week's Questions/Goals

- Follow procedure to get creation/annihilation operators for the other equations.
  - I think after you've gotten it for the complex-valued Klein-Gordon quantum field, you've got it for all cases. (I will have to check that the result agrees with the Schrödinger mechanics case.)
- What is the position-eigenstate creation operator for any mechanics[1]? Is it $\phi^*(x,t)$ in all cases? What about mixing momentum-eigenstate creation and annihilation operators?
- How do QFT states evolve? (exact, perturbative, and numerical approaches)
- What's so bad about negative energies? In an inverse-square potential, there are infinite energies. If we write out the equation of motion for that system, will we be shocked to find negative energies?
- How do things work in the Schrödinger picture? (With and without(?) functional analysis?)
- From fellow student:
  - ... That doesn't make sense; if the creation operator is $\hat{\varphi}^*(x,t)$ (for the Schrödinger equation), then your states won't be normalized! (I think that's what he said.)
  - Why do we quantize the field?
    - My answer: Quantizing the field allows the equation of motion to be applied to states with any number of particles, so it can be applied to QFT states, which have all possible n-particle states.
    - His response: That's half-right.
  - If you haven't gotten to the point of being able to calculate the self-energy of the electron, to say second order in perturbation theory, without using a textbook by the end of this summer, I'd think about going to another department. (And other similar comments.) Also, calculate scattering cross-sections, etc.

[1] Retroactive comment: I meant equation of motion rather than "mechanics".

The Real-Valued Klein-Gordon Quantum Field in the Schrödinger Picture

Procedure for Analysis

(1) Start with equation of motion, Lagrangian, Hamiltonian, conserved quantities

(2) Keep Lorentz invariance and relativistic causality in mind

(3) Derive creation and annihilation operators (make initial interpretation)

(4) Enforce causality on measurements made at spacelike-distant space-time points

(5) Interpret creation and annihilation operators

(6) Interpret field operator actions

(7) Determine conserved quantities and what they represent

(interpretations may have to wait until a gauge field is added and we compare with the complex-valued Klein-Gordon quantum field)

Analysis

(1) Start with equation of motion, Lagrangian, Hamiltonian

$$\left(\hat{E}^2 - \left[\hat{\mathbf{p}}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi} \;=\; \left(\hat{p}^2 c^2 - m^2 c^4\right)\hat{\varphi} \;=\; \hat{0}$$

$$-\hbar^2 c^2 \left(\partial^2 + \nu^2\right)\hat{\varphi}(x) \;=\; \hat{0}$$

$$-\left(\hbar^2 {\partial_t}^2 + \left[\mathbf{p}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi}(\mathbf{p}, t) \;=\; \hat{0}$$

where $\hat{E}$, $\hat{\mathbf{p}}$, and $\hat{p}_\mu$ indicate the relativistic energy and momenta operators, $\partial^2 = \Box = \frac{1}{c^2}{\partial_t}^2 - \nabla^2$, $\nu \equiv mc/\hbar$ is the Compton spatial frequency for the particle of mass m, and $\hat{\varphi}(x) = \hat{\varphi}(\mathbf{x}, t)$.

$$\begin{aligned}
\hat{0} &= \left(\hat{p}^2 - m^2\right)\hat{\varphi} \\
\hat{\mathcal{L}}_{\text{K-G}} &= \frac{1}{2}(\partial\hat{\varphi})^2 - \frac{1}{2}m^2\hat{\varphi}^2 \\
\hat{\mathcal{H}}_{\text{K-G}} &= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\hat{\varphi})^2 + \frac{1}{2}m^2\hat{\varphi}^2
\end{aligned}$$

In the Schrödinger Picture

State Kets and Functionals

$$\| \, \Psi(t) \, \rangle\!\rangle = \Big(\big|\Psi_1(t)\big\rangle \, , \big|\Psi_2(t)\big\rangle \, , \ldots , \big|\Psi_n(t)\big\rangle \, , \ldots \Big)$$

Any System (1D, n particles)

$$\begin{aligned} \Big|\Psi_n^f(t)\Big\rangle &= \int \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_n \; f(x_1, x_2, \ldots, x_n, t) \; C^\dagger(x_1) \, C^\dagger(x_2) \cdots C^\dagger(x_n) \; |0\rangle \\ &= \int \mathrm{d}p_1 \, \mathrm{d}p_2 \cdots \mathrm{d}p_n \; \tilde{f}(p_1, p_2, \ldots, p_n, t) \; \tilde{C}^\dagger(p_1) \, \tilde{C}^\dagger(p_2) \cdots \tilde{C}^\dagger(p_n) \; |0\rangle \end{aligned}$$

Schrödinger System (1D, 1 particle with an arbitrary wave-function f)

$$\begin{aligned} \Big|\Psi_1^f(t)\Big\rangle &&= \int \mathrm{d}x \; f(x,t) \; C^\dagger(x) \; |0\rangle \\ &&= \int \mathrm{d}x \; f(x,t) \; \varphi^*(x) \; |0\rangle \\ \Big\langle\phi\Big|\Psi_1^f(t)\Big\rangle \;\equiv\; \Psi_1^f[\phi] &&= \int \mathrm{d}x \; f(x,t) \, \big\langle\phi\big|\varphi^*(x)\big|0\big\rangle \\ &&= \int \mathrm{d}x \; f(x,t) \; \phi^*(x) \, \langle\phi|0\rangle \\ &&= \int \mathrm{d}x \; f(x,t) \; \phi^*(x) \, \Psi_0[\phi] \end{aligned}$$

Klein-Gordon System (1D, 1 particle in an energy-momentum eigenstate with wave-vector k)

$$\begin{aligned} \Big|\Psi_1^k(t)\Big\rangle &= \int \mathrm{d}x \; f_k(x,t) \; C^\dagger(x) \; |0\rangle \\ &\stackrel{?}{=} \int \mathrm{d}x \; f_k(x,t) \; \varphi(x) \; |0\rangle \end{aligned}$$

$$\begin{aligned} \Big\langle\phi\Big|\Psi_1^k(t)\Big\rangle \;\equiv\; \Psi_1^k[\phi](t) &= \int \mathrm{d}x \; f_k(x,t) \, \big\langle\phi\big|\varphi(x)\big|0\big\rangle \\ &\propto \int \mathrm{d}x \; e^{-k \cdot x} \; \phi(x) \, \langle\phi|0\rangle \end{aligned}$$

Hatfield notation (see 10.30):

$$\Psi_1[\phi] \;\; \propto \;\; \int \mathrm{d}x \; e^{-k \cdot x} \; \phi(x) \, \Psi_0[\phi]$$

Klein-Gordon Functional Analysis

$$
\begin{aligned}
\varphi &= \varphi(x) \\
\pi &= \pi(x) \\
0 &= \left[\partial^\mu \partial_\mu + m^2\right] \varphi \\
\mathcal{L}_{\text{K-G}} &= \mathcal{L}_{\text{K-G}}(\varphi, \partial_t \varphi) \\
&= \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 \quad = \quad \frac{1}{2}(\partial^\mu\varphi)(\partial_\mu\varphi) - \frac{1}{2}m^2\varphi^2 \quad = \quad \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}\left|\boldsymbol{\nabla}\varphi\right|^2 - \frac{1}{2}m^2\varphi^2 \\
\mathcal{H}_{\text{K-G}} &= \mathcal{H}_{\text{K-G}}(\varphi, \pi) \\
&= \frac{1}{2}\pi^2 + \frac{1}{2}\left|\boldsymbol{\nabla}\varphi\right|^2 + \frac{1}{2}m^2\varphi^2
\end{aligned}
$$

$$
\pi \quad = \quad \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \quad = \quad \partial_t \varphi \quad \equiv \quad \dot{\varphi}
$$

$$
\begin{aligned}
\left[\varphi(\mathbf{x}), \pi(\mathbf{y})\right] &= \delta(\mathbf{x} - \mathbf{y}) \\
\tfrac{\delta}{\delta\phi(\mathbf{x})}\phi(\mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}) \\
\left[\tfrac{\delta}{\delta\phi(\mathbf{x})}, \varphi(\mathbf{y})\right] &= \delta(\mathbf{x} - \mathbf{y}) \\
\pi(\mathbf{x}) &= -i\tfrac{\delta}{\delta\phi(\mathbf{x})}
\end{aligned}
$$

Schrödinger Functional Analysis

$$
\begin{aligned}
\varphi &= \varphi(x) \\
\pi &= \pi(x) \\
0 &= \left[i\hbar\,\partial_t + \frac{\hbar^2}{2m}{\partial_x}^2 - V \right] \varphi \\
\mathcal{L}_{\text{Schrö}} &= \mathcal{L}_{\text{Schrö}}(\varphi, \varphi^*, \partial_t\varphi, \partial_t\varphi^*) \\
&= \frac{i}{2}\left(\varphi^*\partial_t\varphi - \varphi\,\partial_t\varphi^*\right) - \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) + V(x)\,\varphi^*\varphi \\
\mathcal{H}_{\text{Schrö}} &= \mathcal{H}_{\text{Schrö}}(\varphi, \varphi^*, \pi, \pi^*) \\
&= \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) + V(x)\,\varphi^*\varphi
\end{aligned}
$$

What does $\partial_t\varphi$ mean if we're in the Schrödinger picture? (Must we be in the Heisenberg picture, or can we just condsider it an independent variable $\dot{\varphi}$?)

$$
\begin{aligned}
\pi &= \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} = \frac{i}{2}\varphi^* \qquad \left(\varphi^* = -2i\pi\right) \qquad \text{(Why is there no 1/2 in Hatfield 2.53?)} \\
\pi^* &= \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi^*)} = -\frac{i}{2}\varphi \qquad \left(\varphi = 2i\pi^*\right) \\
H(q,p,t) &= p_i\,\dot{q}_i - L(q,\dot{q},t) = p_i\,\dot{q}_i(q,p,t) - L(q,\dot{q}(q,p,t),t) \\
\partial_t\varphi &\neq \partial_t\varphi(\varphi,\varphi^*,\pi,\pi^*) \qquad \text{(So we'll see that } \partial_t\varphi \text{ disappears in the formula for the Hamiltonian.)} \\
\partial_t\varphi^* &\neq \partial_t\varphi(\varphi,\varphi^*,\pi,\pi^*) \qquad \text{(So we'll see that } \partial_t\varphi^* \text{ disappears in the formula for the Hamiltonian.)} \\
\mathcal{H}_{\text{Schrö}} &= \pi\,\partial_t\varphi + \pi^*\,\partial_t\varphi^* - \mathcal{L}_{\text{Schrö}}(\varphi,\varphi^*,\partial_t\varphi,\partial_t\varphi^*) \\
&= \pi\,\partial_t\varphi + \pi^*\,\partial_t\varphi^* - \frac{i}{2}\left(\varphi^*\partial_t\varphi - \varphi\,\partial_t\varphi^*\right) + \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) - V(x)\,\varphi^*\varphi \\
&= \pi\,\partial_t\varphi + \pi^*\,\partial_t\varphi^* - \left(\pi\,\partial_t\varphi + \pi^*\,\partial_t\varphi^*\right) + \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) - V(x)\,\varphi^*\varphi \\
&= \frac{1}{2}(\partial_x\varphi^*)(\partial_x\varphi) - V(x)\,\varphi^*\varphi \\
&= 2(\partial_x\pi)(\partial_x\pi^*) - 4V(x)\,\pi\pi^* \qquad \text{(right?)} \\
&= -i(\partial_x\pi)(\partial_x\varphi) + 2i\,V(x)\,\pi\varphi \qquad \text{(right?)}
\end{aligned}
$$

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

- Section 3.2 gives a good explanation of why relativity (Lorentz invariance and relativistic causality) leads to antiparticles and creation and annihilation.

[2] Michael E. Peskin, Daniel V. Schroeder: *An Introduction to Quantum Field Theory*, Westview Press (1995)

- Sections 2.1 and 2.4 give a good explanation of why relativity (Lorentz invariance and relativistic causality) leads to antiparticles and creation and annihilation.

[3] Lewis H. Ryder: *Quantum Field Theory, Second Edition*, Cambridge University Press (1996)

- Section 4.2 gives one explanation for why there should be two sets of raising and lowering operators for the complex Klein-Gordon field.

[4] F. Mandl, G. Shaw: *Quantum Field Theory, Revised Edition*, John Wiley & Sons (1993)

[5] I. J. R. Aitchison, A. J. G. Hey: *Gauge Theories in Particle Physics, A Practical Introduction, Third Edition. Volume I: From Relativistic Quantum Mechanics to QED*, Taylor & Francis Group, LLC (2003)

- Section 6.1 gives a good explanation of how to perturbatively examine interacting fields.

[6] Mark Burgess: *Classical Covariant Fields*, Cambridge University Press (2002)

- "This book discusses the classical foundations of field theory, using the language of variational methods and covariance. There is no ofther book which gives such a comprehensive overview of the subject, exploring the limits of what can be achieved with purely classical notions. These classical notions have a deep and important connection with the second quantized field theory, which is shown to follow on from the Schwinger Action Principle. The book takes a pragmatic view of field theory, focusing on issues which are usually omitted from quantum field theory texts. It uses a well documented set of conventions and catalogues results which are often hard to find in the literature. Care is taken to explain how results aise and how to interpret results physically, for graduate students starting out in the field. Many physical examples are profided, making hte book an ideal supplementary text for courses on elementary field theory, group theory, and dynamical systems. It will also be a valuable reference for researchers already workin in these and related areas."
- Some chapter titles: 17. The Schrödinger field, 18. The real Klein-Gordon field, 19. The complex Klein-Gordon field, 20. The Dirac field