Week 5 Lecture: Concepts of Quantum Field Theory (QFT)

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QFT Particles and Creation Operators (in the Heisenberg Picture)

This Week's Questions

- What's the procedure for finding the raising operator for any arbitrary Lagrangian density $\mathcal{L}$ (or Hamiltonian density $\mathcal{H}$, or equation of motion)?
  - It seems that the procedure is to "complete the square" in the Hamiltonian for some set of operators $\hat{a}_n$, that is, make the square $|\hat{a}_n|^2 = \hat{a}_n^\dagger \hat{a}_n = \hat{N}_n$ manifest in the Hamiltonian, and show that $\hat{a}_n^\dagger$ and $\hat{a}_n$ have the properties of creation and annihilation, respectively. The operator $\hat{N}_n$ will thus be scaled to be a number operator for the n^{th} type of particle.

    We "manifest an operator square" and show the operator destroys particles.

    Illustration of this answer starts on page 5 with "Schrödinger Mechanics in the Heisenberg Picture".
  - In Schrödinger mechanics, we find that the coefficients in the energy eigenstate decomposition of the field operator are the annihilation operators $\hat{a}_n(t)$.
  - In Klein-Gordon mechanics, we complete the square just like for the quantum simple harmonic oscillator.
  - Next: Dirac mechanics, "Phi to the fourth" mechanics, ...
- Are the raising operators in the different representations (x or p) Fourier transforms of each other? Are the corresponding wavefunctions also Fourier transforms of each other? (Prove it.)

  Yes, I'm pretty sure.

  First let's straighten out the basics of state representations and notation. For any dimension d:

$$\begin{aligned}
\langle \mathbf{x}|\phi\rangle &\equiv \phi(\mathbf{x}), \\
|\phi\rangle &= \hat{1}\,|\phi\rangle = \left(\int \mathrm{d}^d x\ |\mathbf{x}\rangle\langle\mathbf{x}|\right)|\phi\rangle = \int \mathrm{d}^d x\ \langle\mathbf{x}|\phi\rangle\,|\mathbf{x}\rangle = \int \mathrm{d}^d x\ \phi(\mathbf{x})\,|\mathbf{x}\rangle\,, \\
\langle \mathbf{p}|\phi\rangle &\equiv \phi(\mathbf{p}) \\
&= \langle\mathbf{p}|\left(\int \mathrm{d}^d x\ \phi(\mathbf{x})\,|\mathbf{x}\rangle\right) = \int \mathrm{d}^d x\ \phi(\mathbf{x})\,\langle\mathbf{p}|\mathbf{x}\rangle = (2\pi)^{-d/2}\int \mathrm{d}^d x\ \phi(\mathbf{x})\,e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \tilde{\phi}(\mathbf{p}).
\end{aligned}$$

  We find that $\langle\mathbf{p}|\phi\rangle = \phi(\mathbf{p})$ is the Fourier transform of $\langle\mathbf{x}|\phi\rangle = \phi(\mathbf{x})$; one may make this explicit by using some extra notation, as we have done with the tilde in $\tilde{\phi}(\mathbf{p})$. Note also that

$$|\phi\rangle = \hat{1}\,|\phi\rangle = \left(\int \mathrm{d}^d p\ |\mathbf{p}\rangle\langle\mathbf{p}|\right)|\phi\rangle = \int \mathrm{d}^d p\ \langle\mathbf{p}|\phi\rangle\,|\mathbf{p}\rangle = \int \mathrm{d}^d p\ \phi(\mathbf{p})\,|\mathbf{p}\rangle\,.$$

  Interestingly, in an abstract sense we can claim that $|\mathbf{p}\rangle$ is the Fourier transform of $|\mathbf{x}\rangle$:

$$|\mathbf{p}\rangle = \hat{1}\,|\mathbf{p}\rangle = \left(\int \mathrm{d}^d x\ |\mathbf{x}\rangle\langle\mathbf{x}|\right)|\mathbf{p}\rangle = \int \mathrm{d}^d x\ |\mathbf{x}\rangle\,\langle\mathbf{x}|\mathbf{p}\rangle = (2\pi)^{-d/2}\int \mathrm{d}^d x\ |\mathbf{x}\rangle\,e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

  Now, if there exist a point-particle creation operator $C^\dagger(\mathbf{x},t)$ and a momentum-eigen-particle creation operator $a^\dagger(\mathbf{p},t)$, so

$$\begin{aligned}
C^\dagger(\mathbf{x},t)\,|0\rangle &= |\mathbf{x}\rangle\,, \\
a^\dagger(\mathbf{p},t)\,|0\rangle &= |\mathbf{p}\rangle\,,
\end{aligned}$$

then we see that $a^\dagger(\mathbf{p},t)\,|0\rangle$ is the Fourier transform of $C^\dagger(\mathbf{x},t)\,|0\rangle$. To see if this holds for the operators themselves, we need to see if this relation holds for a general state. Since I'm not sure how to handle a fully general QFT state yet, I'll just use this simpler "proof":

$$\begin{aligned}
a^\dagger(\mathbf{p},t)\,|0\rangle &= |\mathbf{p}\rangle = (2\pi)^{-d/2}\int \mathrm{d}^d x\ |\mathbf{x}\rangle\ e^{-i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^{-d/2}\int \mathrm{d}^d x\ C^\dagger(\mathbf{x},t)\,|0\rangle\ e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \left((2\pi)^{-d/2}\int \mathrm{d}^d x\ C^\dagger(\mathbf{x},t)\,e^{-i\mathbf{p}\cdot\mathbf{x}}\right)|0\rangle \\
\Rightarrow \qquad a^\dagger(\mathbf{p},t) &= (2\pi)^{-d/2}\int \mathrm{d}^d x\ C^\dagger(\mathbf{x},t)\,e^{-i\mathbf{p}\cdot\mathbf{x}}
\end{aligned}$$

So $a^\dagger(\mathbf{p},t)$ is, apparently, in an abstract sense, the Fourier transform of $C^\dagger(\mathbf{x},t)$.

What about the wavefunctions in these n-particle expressions:

$$\begin{aligned}
|\Psi_n\rangle &= \int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\cdots\mathrm{d}^d x_n\ f(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\ C^\dagger(\mathbf{x}_1,t)\,C^\dagger(\mathbf{x}_2,t)\cdots C^\dagger(\mathbf{x}_n,t)\ |0\rangle \\
&= \int \mathrm{d}^d p_1\,\mathrm{d}^d p_2\cdots\mathrm{d}^d p_n\ g(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n,t)\ \tilde{C}^\dagger(\mathbf{p}_1,t)\,\tilde{C}^\dagger(\mathbf{p}_2,t)\cdots\tilde{C}^\dagger(\mathbf{p}_n,t)\ |0\rangle\,.
\end{aligned}$$

Is g the Fourier transform of f? First let's establish this[1]:

$$\begin{aligned}
C^\dagger(\mathbf{x},t)\,|0\rangle &= |\mathbf{x}\rangle \\
\left\langle \mathbf{x}\middle|C^\dagger(\mathbf{x}',t)\middle|0\right\rangle &= \langle\mathbf{x}|\mathbf{x}'\rangle = \delta(\mathbf{x}-\mathbf{x}') \\
C^\dagger(\mathbf{x}_1,t)\,C^\dagger(\mathbf{x}_2,t)\cdots C^\dagger(\mathbf{x}_n,t)\,|0\rangle &= |\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle \\
\left\langle \mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\middle|C^\dagger(\mathbf{x}_1',t)\,C^\dagger(\mathbf{x}_2',t)\cdots C^\dagger(\mathbf{x}_n',t)\middle|0\right\rangle &= \langle\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n|\mathbf{x}_1',\mathbf{x}_2',\ldots,\mathbf{x}_n'\rangle \\
&= \delta(\mathbf{x}_1-\mathbf{x}_1')\,\delta(\mathbf{x}_2-\mathbf{x}_2')\cdots\delta(\mathbf{x}_n-\mathbf{x}_n')
\end{aligned}$$

$$\begin{aligned}
\langle\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n|\Psi_n\rangle &= \int \mathrm{d}^d x_1'\,\mathrm{d}^d x_2'\cdots\mathrm{d}^d x_n'\ f(\mathbf{x}_1',\mathbf{x}_2',\ldots,\mathbf{x}_n',t)\ \times \\
&\qquad \left\langle \mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\middle|C^\dagger(\mathbf{x}_1',t)\,C^\dagger(\mathbf{x}_2',t)\cdots C^\dagger(\mathbf{x}_n',t)\middle|0\right\rangle \\
&= \int \mathrm{d}^d x_1'\,\mathrm{d}^d x_2'\cdots\mathrm{d}^d x_n'\ f(\mathbf{x}_1',\mathbf{x}_2',\ldots,\mathbf{x}_n',t)\ \delta(\mathbf{x}_1-\mathbf{x}_1')\,\delta(\mathbf{x}_2-\mathbf{x}_2')\cdots\delta(\mathbf{x}_n-\mathbf{x}_n') \\
&= f(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t) \\
\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\Psi_n\rangle &= g(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n,t).
\end{aligned}$$

Now we can simply do as before:

$$\begin{aligned}
g(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n,t) &= \langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\Psi_n\rangle \\
&= \langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\left(\int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\cdots\mathrm{d}^d x_n\ f(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\ |\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle\right) \\
&= \int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\cdots\mathrm{d}^d x_n\ f(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\ \langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n\rangle \\
&= (2\pi)^{-nd/2}\int \mathrm{d}^d x_1\,\mathrm{d}^d x_2\cdots\mathrm{d}^d x_n\ f(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\ \times \\
&\qquad \exp\left(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2+\cdots+\mathbf{p}_n\cdot\mathbf{x}_n\right) \\
&= \tilde{f}(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n,t)
\end{aligned}$$

[1] I may be neglecting important details about whether we are dealing with bosons, fermions, or other kinds of particles.

Additional Questions and Observations

- $\hat{\varphi}^*$ *should* be written $\hat{\varphi}^\dagger$, right? (We just write $\hat{\varphi}^*$ to emphasize that the eigen-fields are complex-valued, i.e., the field eigenvalues are complex.)
- Once we know what $\hat{\varphi}(x,t)$ "means" we should ask, what does $\hat{\pi}(x,t)$ "mean"?
  - This relates to the general question, what does a generalized momentum represent (given that you know what the generalized coordinate represents)? I'll try to give an answer: a coordinate in flux tends to stay in flux unless acted upon by a generalized force, and the generalized momentum is a kind of measure of the "quantity of flux" (like Newton's "quantity of motion", momentum) and the generalized force is the rate of change of that "quantity of flux".

    For fields, we might say, a field (at a certain location) ϕ in flux tends to stay in flux unless acted upon by a generalized "field force" (which can come from the same field at a location nearby), which changes the generalized momentum π.
- It appears that in QFT, the $\hat{x}$ and $\hat{p}$ operators still exist and are used. One just has to be careful about distinguishing between the operators and the variables (which are sometimes seen as mere indices on operators). Take the Klein-Gordon equation, for example:

$$\left(\hat{E}^2 - \left[\hat{\mathbf{p}}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi} = \hat{0}.$$

  In the "space-time" representation, we have

$$\left(-\hbar^2\,\partial_t{}^2 - \left[-\hbar^2 \boldsymbol{\nabla}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi}(\mathbf{x},t) = \hat{0},$$

  but in the "momentum-time" representation, we have

$$\left(-\hbar^2\,\partial_t{}^2 - \left[\mathbf{p}^2 c^2 + m^2 c^4\right]\right)\hat{\varphi}(\mathbf{p},t) = \hat{0}.$$

  I suppose the major problem is that in relativity, space and time are supposed to be on semi-equal footing, but there is no time operator.

  A question I have is, how can I take the general operator version of the Klein-Gordon equation (the first version above) and use kets or bras to get the second two versions?
  - Note that $\hat{\varphi}^*(\mathbf{x},t) = \hat{\varphi}(\mathbf{x},t)$ so that eigenstates $\phi(\mathbf{x},t)$ are real. (How do we know from the start that they should be real?)
  - Note also that $\hat{\varphi}(\mathbf{p},t)$ is the Fourier transform of $\hat{\varphi}(\mathbf{x},t)$, so its eigenstates $\phi^*(\mathbf{p},t)$ are complex and $\hat{\varphi}^*(\mathbf{p},t) = \hat{\varphi}(-\mathbf{p},t)$.

Note This Usual Assumption

We assume, in the Schrödinger picture, that $H \neq H(t)$ and that the energy eigenstates are not degenerate (otherwise $a^\dagger(t)\,|0\rangle$ would be ambiguous).

In the Heisenberg Picture

$$
\begin{aligned}
\| \Psi \rangle\!\rangle &= \big(|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle, \ldots\big) \\
&\neq \| \Psi(t) \rangle\!\rangle \\
\varphi &= \varphi(x,t) \\
\dot{\varphi} &\equiv \partial_t \varphi \quad = \quad \partial_t \varphi(x,t) \\
\pi &= \pi(x,t)
\end{aligned}
$$

I'm not sure, but I think that even though you see time-dependence in the integrand, the integral is time-independent:

$$
\begin{aligned}
\left|\Psi_n^f\right\rangle &= \int \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n\; f(x_1, x_2, \ldots, x_n, t)\; a^\dagger(x_1,t)\, a^\dagger(x_2,t) \cdots a^\dagger(x_n,t)\; |0\rangle \\
&= \int \mathrm{d}p_1\,\mathrm{d}p_2 \cdots \mathrm{d}p_n\; \tilde{f}(p_1, p_2, \ldots, p_n, t)\; \tilde{a}^\dagger(p_1,t)\, \tilde{a}^\dagger(p_2,t) \cdots \tilde{a}^\dagger(p_n,t)\; |0\rangle
\end{aligned}
$$

(In the Heisenberg picture, do you have $\hat{A}(t)\,|\psi\rangle = |\psi'\rangle$, where the states $|\psi\rangle$ and $|\psi'\rangle$ are time-independent?)

$$
\begin{aligned}
\left\langle \phi \middle| \Psi_n^f \right\rangle &= \int \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n\; f(x_1, x_2, \ldots, x_n, t) \left\langle \phi \middle| \hat{a}^\dagger(x_1,t)\, \hat{a}^\dagger(x_2,t) \cdots \hat{a}^\dagger(x_n,t) \middle| 0 \right\rangle \\
&\stackrel{?}{=} \int \mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_n\; f(x_1, x_2, \ldots, x_n, t)\; a^*(x_1,t)\, a^*(x_2,t) \cdots a^*(x_n,t)\, \langle \phi | 0 \rangle
\end{aligned}
$$

It appears that ϕ *is* a wave-function, and here we are finding the overlap (the inner product) between f and ϕ, or the probability that the state $\left|\Psi_n^f\right\rangle$ (or f) is in (or is "measured" to be in) the state $|\phi\rangle$ (or ϕ).

So it looks like we're expanding $\left|\Psi_n^f\right\rangle$ in terms of arbitrary wave-functions (like we would expand in a basis, such as $|x\rangle$, $|p\rangle$, or energy eigenstates), but the set of arbitrary wave-functions is a *much* larger "basis", and they're not necessarily independent wave-functions. . . (only if they're not additive in the usual sense would I expect them to be independent). We're getting into functional analysis here.

But what is the number $\langle \phi | 0 \rangle$? How much can a state $|\phi\rangle$ overlap with the vacuum state $|0\rangle$?

Mustn't $\langle \phi | 0 \rangle$ be equal to 1?

Do divergence problems crop up with $\langle \phi | 0 \rangle$?

Schrödinger Mechanics in the Heisenberg Picture: creation/annihilation operators

In quantum field theory (QFT), we promote the fields $\phi(x,t)$ (a.k.a. wave-functions in quantum mechanics) to operator fields $\hat{\varphi}(x,t)$ that operate on QFT states $\| \Psi \rangle\!\rangle$. In the QFT of (1+1)-D Schrödinger mechanics, these fields must obey the Schrödinger field equation

$$\left[i\,\partial_t + \frac{1}{2}{\partial_x}^2 - V(x) \right] \hat{\varphi}(x,t) = \hat{0}$$

and their eigenstates are (sets of?) fields that obey this equation. We'll find out that these eigenstate fields are, in fact, (specially normalized?) wave-functions. [2] We assume[3] that the field operators $\hat{\varphi}$ can be decomposed into operators corresponding to energy eigenstates, similar to how regular wave-functions can be decomposed into energy eigenstates of the usual one-particle Hamiltonian h:

$$\begin{aligned}
\hat{\varphi}(x,t) &= \sum_n \hat{a}_n(t)\,\phi_n(x) \\
\hat{\varphi}^*(x,t) &= \sum_n \hat{a}^\dagger_n(t)\,\phi^*_n(x) \\
\hat{h}(x)\,\phi_n(x) &= e_n\,\phi_n(x).
\end{aligned}$$

We choose to make the coefficients $a_n(t)$ and $a^\dagger_n(t)$ operators and leave the eigenfunctions as functions. [4] With this assumption, we can rearrange the Hamiltonian of Schrödinger mechanics into something familiar. Using normal ordering (to avoid divergences later on in the theory),

$$\begin{aligned}
\hat{H}_{\text{Schrö}}(t) &= \int \mathrm{d}x\ \hat{\mathcal{H}}_{\text{Schrö}}\big(\hat{\varphi}(x,t), \hat{\varphi}^*(x,t)\big) \\
&= \int \mathrm{d}x \left[\frac{1}{2}(\partial_x\hat{\varphi}^*)(\partial_x\hat{\varphi}) + V(x)\,\hat{\varphi}^*\hat{\varphi} \right] \\
&= \frac{1}{2}\left[\hat{\varphi}^*\partial_x\hat{\varphi}\right]_{\text{bndry}} + \int \mathrm{d}x \left[-\hat{\varphi}^*\frac{1}{2}{\partial_x}^2\hat{\varphi} + \hat{\varphi}^*\,V(x)\,\hat{\varphi} \right] \\
&= 0 + \int \mathrm{d}x\ \hat{\varphi}^* \left(-\frac{1}{2}{\partial_x}^2 + V(x) \right) \hat{\varphi} \\
&= \int \mathrm{d}x\ \hat{\varphi}^*\,\hat{h}\,\hat{\varphi} \\
&= \int \mathrm{d}x \left(\sum_n \hat{a}^\dagger_n(t)\,\phi^*_n \right) \hat{h} \left(\sum_m \hat{a}_m(t)\,\phi_m \right) \\
&= \sum_{nm} \hat{a}^\dagger_n(t)\ \hat{a}_m(t) \int \mathrm{d}x\ \phi^*_n\,\hat{h}\,\phi_m \\
&= \sum_{nm} \hat{a}^\dagger_n(t)\ \hat{a}_m(t)\ \int \mathrm{d}x\ \phi^*_n\,e_m\,\phi_m \\
&= \sum_{nm} \hat{a}^\dagger_n(t)\ \hat{a}_m(t)\ e_m\,\delta_{nm} \\
&= \sum_n e_n\ \hat{a}^\dagger_n(t)\ \hat{a}_n(t).
\end{aligned}$$

[2] I'll have to work on this explanation, but the trick is that all particles and any collection of particles must obey this equation, and it's the field operator that "talks to" every possible combination of particles and determines their behavior.

[3] I think this assumption implies that we are only considering bound states or that our system is finite, as in a box with periodic boundary conditions.

[4] One should investigate the other option too.

This is like the Hamiltonian for the quantum simple harmonic oscillator (SHO), but now we have an infinite set of independent "oscillators" with their own energy quanta e_n. We shall see each quantum is a particle. We also have

$$\begin{aligned}
\sum_n \phi_n^*(x')\,\phi_n(x) &= \delta(x-x') \\
\text{and} \qquad -i\left[\varphi(x,t),\pi(x',t)\right] &= \left[\varphi(x,t),\varphi^*(x',t)\right] \\
&= \left[\sum_n \hat{a}_n(t)\,\phi_n(x), \sum_m \hat{a}_m^\dagger(t)\,\phi_m^*(x')\right] \\
&= \sum_{nm}\left[\hat{a}_n(t),\hat{a}_m^\dagger(t)\right]\phi_n(x)\,\phi_m^*(x') \\
&= \delta(x-x') \\
\Rightarrow \qquad \left[\hat{a}_n(t),\hat{a}_m^\dagger(t)\right] &= \delta_{nm}.
\end{aligned}$$

From here we can (but won't in this document) use the same procedure as in the case of the 1D SHO of Schrödinger mechanics to show that $\hat{a}_n(t)$ and $\hat{a}_n^\dagger(t)$ behave much like the annihilation and creation operators in that case. The operator $\hat{a}_n^\dagger(t)$ takes an "integer-number-eigen-particle" state[5]

$$\|\,\Psi\,\rangle\!\rangle = (m_1, m_2, \ldots, m_n, \ldots),$$

$$\text{or} \qquad \|\,\Psi\,\rangle\!\rangle = \begin{pmatrix} 1 & 2 & \cdots & n & \cdots \\ m_1 & m_2 & \cdots & m_n & \cdots \end{pmatrix},$$

time-evolves it from the reference time t_0 to time t, adds a "particle" or quantum of energy e_n into the system (distributed over space however that unique eigenstate is distributed), and then time-deëvolves the state back to the reference time and multiplies by a constant, so[6]

$$\hat{a}_n^\dagger(t)\,\|\,\Psi\,\rangle\!\rangle \propto (m_1, m_2, \ldots, m_n+1, \ldots),$$

$$\hat{a}_n^\dagger(t)\,|0\rangle = |\phi_n\rangle = |n\rangle\,.$$

The operator $\hat{a}_n(t)$ evolves $|\psi\rangle$ to time t, destroys or takes away a particle of energy e_n if any such particles already exist in the system, or returns the null vector if no such particles exist in the system, then deëvolves back to the reference time and multiplies by some constant.

$$\hat{a}_n(t)\begin{pmatrix} 1 & 2 & \cdots & n & \cdots \\ m_1 & m_2 & \cdots & 0 & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n & \cdots \\ m_1 & m_2 & \cdots & |\rangle & \cdots \end{pmatrix} \qquad \text{or} \qquad 0?$$

$$\hat{a}_n(t)\,|0\rangle = |\rangle\,,$$

$$\hat{a}_n^\dagger(t)\,\hat{a}_n(t)\,\|\,\Psi\,\rangle\!\rangle = m_n\,(m_1, m_2, \ldots, m_n, \ldots).$$

Now the question is, what do the operators $\varphi(x,t)$ and $\varphi^*(x,t)$ do? We can use the wonderful properties

[5] See "Possible Notation" below.

[6] I should probably write this as $(\hat{1}, \hat{1}, \ldots, \hat{a}_n^\dagger, \ldots)\,\|\,\Psi\,\rangle\!\rangle$.

of the energy eigenstates to find out.

$$\begin{aligned}
\int \mathrm{d}x\ \phi_n^*(x)\,\phi_m(x) &= \delta_{nm} \\
\int \mathrm{d}x\ \phi_n^*(x)\,\hat{\varphi}(x,t) &= \int \mathrm{d}x\ \phi_n^*(x)\sum_m \hat{a}_m(t)\,\phi_m(x) \\
&= \sum_m \hat{a}_m(t)\int \mathrm{d}x\ \phi_n^*(x)\,\phi_m(x) \\
&= \sum_m \hat{a}_m(t)\,\delta_{nm} \\
&= \hat{a}_n(t) \\
\int \mathrm{d}x\ \phi_n(x)\,\hat{\varphi}^*(x,t) &= \int \mathrm{d}x\ \phi_n(x)\sum_m \hat{a}_m^\dagger(t)\,\phi_m^*(x) \\
&= \sum_m \hat{a}_m^\dagger(t)\int \mathrm{d}x\ \phi_n(x)\,\phi_m^*(x) \\
&= \sum_m \hat{a}_m^\dagger(t)\,\delta_{nm} \\
&= \hat{a}_n^\dagger(t)
\end{aligned}$$

Given the last set of relations and given that $\hat{a}_n^\dagger(t)$ creates a quantum of energy e_n, we have

$$\begin{aligned}
\hat{a}_n^\dagger(t)\,|0\rangle &= |\phi_n\rangle \\
&= \int \mathrm{d}x\ \phi_n(x)\,|x\rangle \\
\hat{a}_n^\dagger(t)\,|0\rangle &= \int \mathrm{d}x\ \phi_n(x)\,\hat{\varphi}^*(x,t)\,|0\rangle \\
\Rightarrow \qquad \hat{\varphi}^*(x,t)\,|0\rangle &= |x\rangle\,,
\end{aligned}$$

or, if you prefer,

$$\begin{aligned}
\left\langle x\middle|\hat{a}_n^\dagger(t)\middle|0\right\rangle &= \langle x|\phi_n\rangle = \phi_n(x) \\
&= \left\langle x\middle|\hat{a}_n^\dagger(t)\middle|0\right\rangle \\
&= \left\langle x\middle|\int \mathrm{d}x'\ \phi_n(x')\,\hat{\varphi}^*(x',t)\middle|0\right\rangle \\
&= \int \mathrm{d}x'\ \phi_n(x')\,\left\langle x\middle|\hat{\varphi}^*(x',t)\middle|0\right\rangle \\
&= \phi_n(x) \\
\Rightarrow \qquad \left\langle x\middle|\hat{\varphi}^*(x',t)\middle|0\right\rangle &= \delta(x-x') \\
\Rightarrow \qquad \hat{\varphi}^*(x',t)\,|0\rangle &= \left|x'\right\rangle\,,
\end{aligned}$$

since $\left\langle x\middle|x'\right\rangle = \delta(x-x')$. So $\hat{\varphi}^*(x,t)$ is a point-particle creation operator that creates a particle at the location x. When $\hat{\varphi}^*(x,t)$ is integrated and weighted with a wave-function $\phi_n(x)$, the integral is the operator that produces the state with that wave-function, $|\phi_n\rangle$.

And what does $\hat{\varphi}(x,t)$ do? If we assume

$$\hat{\varphi}(x,t)\left|x'\right\rangle = \delta(x-x')\,|0\rangle\,,$$

then it will do what it should:

$$
\begin{aligned}
\hat{a}_n(t)\,|\phi_n\rangle &= |0\rangle \\
&\neq \int \mathrm{d}x\ 0 * |x\rangle \quad = \quad |\rangle \\
\hat{a}_n(t)\,|\phi_n\rangle &= \left(\int \mathrm{d}x\ \phi_n^*(x)\,\hat{\varphi}(x,t)\right)\left(\int \mathrm{d}x'\ \phi_n(x')\,|x'\rangle\right) \\
&= \int \mathrm{d}x\ \phi_n^*(x) \int \mathrm{d}x'\ \phi_n(x')\ \hat{\varphi}(x,t)\,|x'\rangle \\
&= \int \mathrm{d}x\ \phi_n^*(x) \int \mathrm{d}x'\ \phi_n(x')\ \delta(x-x')\,|0\rangle \\
&= \int \mathrm{d}x\ \phi_n^*(x)\,\phi_n(x)\,|0\rangle \\
&= |0\rangle \\
\Rightarrow \qquad \hat{\varphi}(x,t)\,|x'\rangle &= \delta(x-x')\,|0\rangle\,.
\end{aligned}
$$

To be complete, let's look at this too:

$$
\begin{aligned}
\hat{\varphi}(x,t)\,|x\rangle &= \hat{\varphi}(x,t)\int \mathrm{d}x'\delta(x-x')\,|x'\rangle \\
&= \int \mathrm{d}x'\delta(x-x')\,\hat{\varphi}(x,t)\,|x'\rangle \\
&= \int \mathrm{d}x'\delta(x-x')\,\delta(x-x')\,|0\rangle \\
&= \int \mathrm{d}x'\delta(x-x')^2\,|0\rangle\,.
\end{aligned}
$$

What's going on here? Let's just say for now that the operator $\hat{\varphi}(x,t)$ destroys or annihilates point particles and can destroy any state by any amount via integration with a weighting wave-function.

Klein-Gordon Mechanics in the Heisenberg Picture: creation/annihilation operators

Now we examine the $(3+1)$-D QFT of Klein-Gordon mechanics determined by

$$\left[\partial^\mu \partial_\mu + m^2\right] \hat{\varphi}(\mathbf{x}, t) = \hat{0}.$$

This time the Hamiltonian takes this form[7]:

$$\begin{aligned} \hat{H}_{\text{K-G}}(t) &= \int \mathrm{d}^3x \; \hat{\mathcal{H}}_{\text{K-G}}\big(\hat{\varphi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}, t)\big) \\ &= \int \mathrm{d}^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left|\boldsymbol{\nabla}\hat{\varphi}\right|^2 + \frac{1}{2}m^2\hat{\varphi}^2\right]. \end{aligned}$$

There's no apparent way to manifest an operator square using this form, so we try another form in the momentum representation[8]:

$$\begin{aligned} \hat{H}_{\text{K-G}}(t) &= \int \mathrm{d}^3p \; \hat{\mathcal{H}}_{\text{K-G}}\big(\hat{\varphi}(\mathbf{p}, t), \hat{\pi}(\mathbf{p}, t)\big) \\ &= \int \mathrm{d}^3p \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left|i\mathbf{p}\hat{\varphi}\right|^2 + \frac{1}{2}m^2\hat{\varphi}^2\right] \\ &= \int \mathrm{d}^3p \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\mathbf{p}^2\hat{\varphi}^2 + \frac{1}{2}m^2\hat{\varphi}^2\right] \\ &= \int \mathrm{d}^3p \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\mathbf{p}^2 + m^2\right)\hat{\varphi}^2\right] \end{aligned}$$

Ooh. It looks easy now, since it's in the form $\hat{A}^2 + \hat{B}^2$. Let's define ${e_{\mathbf{p}}}^2 \equiv \mathbf{p}^2 + m^2$, so

$$\hat{H}_{\text{K-G}}(t) = \int \mathrm{d}^3p \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}{e_{\mathbf{p}}}^2\hat{\varphi}^2\right]$$

Now this is precisely the same situation as in the quantum SHO, except there are a continuum of independent "oscillators" each with their own energy quanta $e_{\mathbf{p}}$. We'll make the same definitions as for the SHO, but now they depend on $\mathbf{p}$:

$$\hat{a}^\dagger_{\mathbf{p}}(t) \equiv \sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\hat{\varphi}(\mathbf{p}, t) - i\frac{\hat{\pi}(\mathbf{p}, t)}{e_{\mathbf{p}}}\right) \qquad \hat{a}_{\mathbf{p}}(t) \equiv \sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\hat{\varphi}(\mathbf{p}, t) + i\frac{\hat{\pi}(\mathbf{p}, t)}{e_{\mathbf{p}}}\right)$$

so

$$\hat{\varphi}(\mathbf{p}, t) = \frac{1}{\sqrt{2e_{\mathbf{p}}}}\left(\hat{a}^\dagger_{\mathbf{p}}(t) + \hat{a}_{\mathbf{p}}(t)\right) \qquad \hat{\pi}(\mathbf{p}, t) = i\sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\hat{a}^\dagger_{\mathbf{p}}(t) - \hat{a}_{\mathbf{p}}(t)\right).$$

Thus we have

$$\hat{H}_{\text{K-G}}(t) = \int \mathrm{d}^3p \; e_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}(t)\,\hat{a}_{\mathbf{p}}(t) + \frac{1}{2}\right)$$

and we get rid of the 1/2, because differences in energy are all that matter after all.

$$\hat{H}_{\text{K-G}}(t) = \int \mathrm{d}^3p \; e_{\mathbf{p}} \; \hat{a}^\dagger_{\mathbf{p}}(t)\,\hat{a}_{\mathbf{p}}(t)$$

[7] Somehow we know that $\phi(x, t)$ and $\pi(x, t)$ should be real, so the Hamiltonian does not have $\hat{\varphi}^*$ or $\hat{\pi}^*$ as arguments. How do we know that, initially?

[8] We could have used this representation in the Schrödinger mechanics, but it wouldn't have helped us.

So we have that $\hat{a}^\dagger_{\mathbf{p}}(t)$ creates a particle of momentum $\mathbf{p}$ (which is actually an unphysical state that exists everywhere with equal probability). [I should explain this creation more.]

$$\begin{aligned} \hat{a}^\dagger_{\mathbf{p}}(t)\,|0\rangle &= \left|\phi_{\mathbf{p}}\right\rangle = |\mathbf{p}\rangle \\ \left\langle \mathbf{x}\middle|\hat{a}^\dagger_{\mathbf{p}}(t)\middle|0\right\rangle &= \langle\mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}}e^{-i\mathbf{p}\cdot\mathbf{x}} \end{aligned}$$

Such a creation operator is most useful in creating physical states by integration with a weighting factor:

$$\begin{aligned} |\mathbf{x}\rangle &= \int \mathrm{d}^3p\ |\mathbf{p}\rangle\langle\mathbf{p}|\mathbf{x}\rangle = \int \mathrm{d}^3p\ \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3p\ e^{i\mathbf{p}\cdot\mathbf{x}}\ \hat{a}^\dagger_{\mathbf{p}}(t)\,|0\rangle \\ |\phi\rangle &= \int \mathrm{d}^3x\ |\mathbf{x}\rangle\langle\mathbf{x}|\phi\rangle = \int \mathrm{d}^3x\ \phi(\mathbf{x})\,|\mathbf{x}\rangle = \int \mathrm{d}^3x\ \phi(\mathbf{x})\left(\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3p\ e^{i\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{p}\rangle\right) \\ &= \int \mathrm{d}^3p\left(\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3x\ \phi(\mathbf{x})\,e^{i\mathbf{p}\cdot\mathbf{x}}\right)|\mathbf{p}\rangle = \int \mathrm{d}^3p\ \tilde{\phi}(\mathbf{p})\,|\mathbf{p}\rangle \\ &= \int \mathrm{d}^3p\ \tilde{\phi}(\mathbf{p})\ \hat{a}^\dagger_{\mathbf{p}}(t)\,|0\rangle \end{aligned}$$

$$\begin{aligned} C^\dagger(\mathbf{p})\,|0\rangle &= |\mathbf{p}\rangle \\ \Rightarrow\qquad C^\dagger(\mathbf{p}) &= \hat{a}^\dagger_{\mathbf{p}}(t) \\ C^\dagger(\mathbf{x})\,|0\rangle &= |\mathbf{x}\rangle \\ \Rightarrow\qquad C^\dagger(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3p\ e^{i\mathbf{p}\cdot\mathbf{x}}\ \hat{a}^\dagger_{\mathbf{p}}(t) \\ &= \tilde{C}^\dagger(\mathbf{p}) \\ &= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3p\ e^{i\mathbf{p}\cdot\mathbf{x}}\ \sqrt{\frac{e_{\mathbf{p}}}{2}}\left(\hat{\varphi}(\mathbf{p},t) - i\frac{\hat{\pi}(\mathbf{p},t)}{e_{\mathbf{p}}}\right) \\ &= \cdots \end{aligned}$$

maybe use the real-ness of the fields $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ to simplify…what do $\hat{\varphi}(\mathbf{p},t)$ and $\hat{\pi}(\mathbf{p},t)$ do?

Potential Notation

We could call { $|\Psi\rangle$, etc. } the Dirac bra-ket notation and { $\|\Psi\rangle\!\rangle$, etc. } the Fock bra-ket notation.

- How shall the following states be normalized? ...
- In terms of n-particle states (of any mechanics and system):

$$\begin{aligned}
\|\Psi\rangle\!\rangle &= |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\
&= \text{``}\,|\Psi_1\rangle + |\Psi_2\rangle + \cdots + |\Psi_n\rangle + \cdots\text{''} \qquad \text{(lazy notation for sketchy work)} \\
&= \big(|\Psi_1\rangle\,, |\Psi_2\rangle\,, \ldots, |\Psi_n\rangle\,, \ldots\big) \\
&= (\Psi_1, \Psi_2, \ldots, \Psi_n, \ldots) \qquad \text{(also lazy notation)} \\
&= \begin{pmatrix} 1 & 2 & 7 & n \\ f & g & s & t \end{pmatrix} = \begin{pmatrix} \text{number of particles } n \\ n\text{-particle wave-function} \end{pmatrix} \qquad \text{(ignoring all zero-functions)} \\
&= \|\Psi_1\rangle\!\rangle + \|\Psi_2\rangle\!\rangle + \cdots + \|\Psi_n\rangle\!\rangle + \cdots \\
\|\Psi_n\rangle\!\rangle &= |0\rangle \oplus |0\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\
&= \begin{pmatrix} n \\ f \end{pmatrix}
\end{aligned}$$

- In terms of energy eigenstates (of a particular mechanics and system):

$$\begin{aligned}
\|\Psi\rangle\!\rangle &= \begin{pmatrix} 1 & 2 & 7 & n \\ 2 & 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} \text{energy eigenstate index } n \\ \text{number of particles } m_n \end{pmatrix} \qquad \text{(ignoring all unoccupied eigenstates)} \\
&= (2, 2, 0, 0, 0, 0, 1, 0, 0, \ldots, 0, 5, 0, 0, \ldots) \\
&= (m_1, m_2, \ldots, m_n, \ldots) \\
\langle\!\langle\Psi\|\Psi\rangle\!\rangle &\stackrel{?}{=} \sum_n m_n
\end{aligned}$$

One could call the state above an "integer-number-eigen-particle" state.

$$\left\| \Psi_n^f(t) \right\rangle\!\!\rangle = \int \mathrm{d}^3x_1\,\mathrm{d}^3x_2 \cdots \mathrm{d}^3x_n \; f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, t)\; \mathbb{C}^\dagger(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, t)\; \|\, 0\,\rangle\!\rangle$$

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)