Week 4 Lecture: Concepts of Quantum Field Theory (QFT)

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QFT States and Wavefunctions

This Week's Questions

- In QFT we have fields, but where are the particles? Where are the wavefunctions? Is the wavefunction an eigenvalue of the field operator? (In particular, how do we get regular quantum mechanics out of QFT?)
- What does a QFT state "look" like? Is it a particular configuration of the field? Are there eigenstates with eigenvalue fields? (What would a plot of a QFT state in position-representation look like? Can it be done?)
- For free field theory, what is the equation of motion for the states and wavefunctions?
- For an interacting field theory, what is the equation of motion for the states and wavefunctions?

These questions are so natural and fundamental to QFT that one would expect that they would be answered within the first week or so of an introductory course on QFT and that anyone researching advanced physics would be able to give a correct and relatively succinct answer to each of them. The reality as of 2008 does not meet these expectations.

After looking through many books on QFT, one book has surpassed all the others in providing answers to most, if not all, of these questions and motivating the answers and procedures. That book is Brian Hatfield's *Quantum Field Theory of Point Particles and Strings* [1]. Perhaps it is not a coincidence that these questions are addressed in a book is ultimately about string theory because string theorists seem to take a different view than those working strictly within QFT for point particles – string quantum field theorists seem to care about these questions more, for some reason, while particle quantum field theorists want to get straight to Feynman diagram calculations.

In this lecture and following lectures, I will present the answers to these questions as I understand them now (along with additional questions that arise), with minimal motivation for now. You should read Hatfield, especially chapters 2 and 10, to get the motivation and deeper explanations. (I will henceforth be developing my own explanations.)

Proposed Answer (a start)

The answers to these questions depend on what "picture" or representation one is using: the Schrödinger, Heisenberg, or interaction (or Dirac) picture. I prefer the Schrödinger picture, since it seems most intuitive, but I'll try to develop the answers in all pictures. Also, I will answer the questions in the most expedient order. In fact, I will generate another question to start with, which is perhaps less natural than the questions listed above.

- What information does a QFT state contain (in the Schrödinger picture)?

QFT, both relativistic and nonrelativistic, aims to describe systems with variable numbers of particles, since in reality particles are created and destroyed. (This, as far as I know, does *not* imply the need for the relativistic framework, as is insinuated by most books on QFT. However, a relativistic QFT, which is Lorentz invariant, will naturally give rise to a new interpretation of the field that involves spin and statistics and antiparticles, and with the antiparticles, variability in particle number.[2] However, we may conceive of other mathematical or physical mechanisms to allow variability in particle number that could be used in a nonrelativistic QFT.) The creation and destruction of particles occurs in two ways, either by

[1] Thanks go out to Professor Per Kraus for recommending this book and giving explanations to various questions.
[2] See Hatfield, pg 35.

exchange with other forms of energy – a particle, as one incarnation of energy, can be exchanged for kinetic energy of another particle, for instance – or by "zero-point" fluctuation in the energy of the system due to uncertainty in its energy.

QFT also aims to retain the probabilistic wavefunction description of particles that quantum mechanics (QM) shows works so well. QM shows that very small particles have some very non-particulate properties so that they might better be called something other than "particles", such as "quanta" or "wavicles", to indicate a different set of notions than the classical ideas of particles. From a quantum mechanical perspective it seems that the wavefunctions are more fundamental than the particles themselves.

To learn how QFT manages to accomplish both of these aims, first we restrict our universe to only one kind of particle (without spin) where the particles do not interact with each other (the particles are "free"). This will be called a "free QFT". So, at any one moment, there can be any number of particles, but the energy of the universe should be fluctuating around one constant value.

Now, I will propose an answer to the question that seems reasonable and consistent with what I know about QFT, and as time goes on I will try to see if this answer is correct. QFT describes a possible state of this universal system with a "super-state" $\| \Psi \rangle\!\rangle$ that contains a state for each possible number of particles.

$$\| \Psi(t) \rangle\!\rangle = \Big(\big|\Psi_1(t)\big\rangle \,, \big|\Psi_2(t)\big\rangle \,, \ldots \,, \big|\Psi_n(t)\big\rangle \,, \ldots \Big)$$

This notation is inspired by the concept of the Fock space (see "Notes" and "Potential Notation" below).

– I think that the equation of motion for a free QFT time-evolves each of these states separately, and as we "turn on" the interactions perturbatively, the equations of motion start to mix the states and allow them to affect each other's evolution.

– We should find in this free QFT that the particle number does not change, unless vacuum fluctuations allow for virtual, temporary particles to exist. We have provided no explicit mechanism yet for the change in particle number.

Notes

- A Fock space $F_s(H)$ (with no potential? so that you can use a single-particle Hamiltonian) is a direct sum all multi-particle Hilbert spaces:

$$F_s(H) = \bigoplus_{n=0}^{\infty} S_s H^{\otimes n},$$

  where H is the single-particle Hamiltonian, s is an index that indicates either bosonic statistics, $s = +$ for bosons, or fermionic statistics, $s = -$ for fermions, S_s is a symmetry operator that either symmetrizes (for $s = +$) or antisymmetrizes (for $s = -$) the tensor product $H^{\otimes n}$.
  $H^{\otimes 0} = \{|0\rangle\}$?
  $H^{\otimes 1} = H$
  $H^{\otimes 2} = H \otimes H$
  $H^{\otimes 3} = H \otimes H \otimes H$, etc.

  (no need for Slater determinant ?)

- In the Heisenberg (Schrödinger?) picture, an n-particle state (is it time-dependent? it looks to be...):

$$\begin{aligned} |\Psi_n\rangle &= \int \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n\; f(x_1, x_2, \ldots, x_n, t)\, \varphi^*(x_1, t)\, \varphi^*(x_2, t) \cdots \varphi^*(x_n, t)\; |0\rangle \\ &= \int \mathrm{d}p_1\, \mathrm{d}p_2 \cdots \mathrm{d}p_n\; \tilde{f}(p_1, p_2, \ldots, p_n, t)\, a^\dagger(p_1, t)\, a^\dagger(p_2, t) \cdots a^\dagger(p_n, t)\; |0\rangle \end{aligned}$$

(see Hatfield equations (2.69) and (2.79)) a general state:

$$\begin{aligned} \| \, \Psi(t) \, \rangle\!\rangle &= \bigoplus_{n=0}^{\infty} \big|\Psi_n(t)\big\rangle \\ &= \big|\Psi_1(t)\big\rangle \oplus \big|\Psi_2(t)\big\rangle \oplus \big|\Psi_3(t)\big\rangle \oplus \cdots \end{aligned}$$

- Schrödinger coordinate representation
  - What does the operator φ "do"? Then, what does $|\phi\rangle$ represent?
    (pg 200) Let $|\phi\rangle$ be an eigenstate of φ with eigenvalue ϕ
    $\varphi(\mathbf{x})\ |\phi\rangle = \phi(\mathbf{x})\ |\phi\rangle$
  - Is $|\Psi\rangle$ a general state?
    (pg 200) The coordinate representation of the state $|\Psi\rangle$, now time dependent, is the wave-functional, $\Psi[\phi]$, a functional of the function $\phi(\mathbf{x})$
    $\Psi[\phi] = \langle\phi|\Psi\rangle$
    For $H \neq H(t)$, we have $\Psi[\phi, t] = e^{-iEt}\,\Psi[\phi]$
  - Why is this not a ket?:
    (pg 208) $\Psi_1[\phi]$ represents a state with 1 scalar particle of mass m, energy ω_{k_1}, and momentum $\mathbf{k}_1$. Since the particle is in a definite state of energy-momentum, then by the uncertainty principle we have no idea where the particle is located. This is reflected in $\Psi_1[\phi]$ by the fact that $\Psi_1[\phi]$ has no dependence on any spatial coordinate x_i.
  - Why is this not a ket?:
    (pg 208) $\eta\,\phi(\mathbf{x}_1)\,\Psi_0[\phi]$ represents a state where the particle is located at x_1. The wave-functional does not depend on any momentum vector so we don't have any idea what the momentum is.
- Vocabulary note for Chris: "dynamical-indistiguishability class" vs. "indistinguishability class"

  Since two electrons that are separated by great distance *are* distinguishable, but we want to define how they are indistinguishable, we may want to say that they are both elements of the same "dynamical indistinguishability class". Two electrons may be spatially distinguishable or kinetically distinguishable, but they are not dynamically distinguishable.

Potential Notation

We could call this a "super ket" or a "Fock ket"

$$\begin{aligned} |\mathbf{\Psi}\rangle \quad \text{or} \quad \|\Psi\rangle\!\rangle &= \bigoplus_{n=0}^{\infty} |\Psi_n\rangle \\ &= |\Psi_1\rangle \oplus |\Psi_2\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\ &= \text{``}\,|\Psi_1\rangle + |\Psi_2\rangle + \cdots + |\Psi_n\rangle + \cdots\text{''} \qquad \text{(lazy notation for sketchy work)} \\ &= \big(|\Psi_1\rangle\,, |\Psi_2\rangle\,, \ldots, |\Psi_n\rangle\,, \ldots\big) \\ &= (\Psi_1, \Psi_2, \ldots, \Psi_n, \ldots) \\ &= \begin{pmatrix} 1 & 2 & 7 & n \\ f & g & s & t \end{pmatrix} \qquad \text{(ignoring all zero-functions)} \\ \|\Psi_n\rangle\!\rangle &= |0\rangle \oplus |0\rangle \oplus \cdots \oplus |\Psi_n\rangle \oplus \cdots \\ &= \begin{pmatrix} n \\ f \end{pmatrix} \end{aligned}$$

Next Steps

- Understand the three pictures of quantum mechanics
- Go back to the bedsprings model and take it one-step-at-a-time up to the new understanding of the QFT state. Proceed as follows:
  - 2 coupled oscillators
  - 3 coupled oscillators
  - ∞ coupled oscillators
  - a continuum of continuously coupled oscillators

  Be sure to distinguish between the meaning of the field and wavefunctions and states. (In each picture they could have different kinds of meanings.)
- Find the QFT kets in the Schrödinger picture
- What's the procedure for finding the raising operator for any arbitrary Lagrangian density $\mathcal{L}$ (or Hamiltonian density $\mathcal{H}$, or equation of motion)?
- Are the raising operators in the different representations (x or p) Fourier transforms of each other? Are the corresponding wavefunctions also Fourier transforms of each other? (Prove it.)

Additional questions

- In QM, the states are normalized to 1, and we either have that the probability of finding a particle integrates to 1 or the number density is 1 (or n for an n-particle state). How are the QFT states normalized, and since the particle number is variable, is it just (proportional to) the expectation value of the energy that is constant (perhaps with some variation due to vacuum fluctuation)? (Show these features of the QFT field explicitly.)

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

  - Chapter 2 is exceptional.