Week 2 Lecture: Concepts of Spin

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Status Report in the Quest to Understand Spin

For reference, here is a list of various properies associated with spin, in small type so as to fit on this page:

Properties associated with spin

- Informal or Anecdotal Property
  - * As one thing rotates or cycles with a period P, another thing (the spinor) with spin S cycles through states with a period P/S. (Sometimes "rotate" is used when "cycles" or "phase-evolves" seem more appropriate.)
- Physical Properties
  - – Angular momentum
    - * Intrinsic angular momentum (as opposed to orbital angular momentum)
    - * Ohanian's paper[1] says this is due to the circulation of energy in the field in question
      (If one is speaking of a particle, then the spin comes from the field of the particle and is not a property that is stipulated mathematically before constructing the field)
    - * Existence of spin related to conservation of angular momentum and Lorentz group invariance (4-spinors emerge naturally in relativistic quantum mechanics or quantum field theory)
  - – Magnetic dipole moment
    - * According to Ohanian, this property arises from the circulation of charge in the field, just like the angular momentum comes from the circulation of energy in the field

      Before reading Ohanian's paper I would have said this property is:
      Not related by the usual relation to angular momentum
      (Cannot be derived by rotation of a solid extended charged object, and does not seem to be able to be described by topological anomalies in spacetime *a la* John Wheeler's geometrodynamics)
    - * Depends on mass! ($\propto m^{-1}$)
  - – Statistics
    - * Pauli exclusion principle and spin-statistics theorem
  - – Potential field (Coupling properties?)
    - * Spin number corresponds to number of indices for potential field (or coupling?)
  - – Radiation field properties
    - * $360°/S$ rotational invariance of classical radiation field of spin S
    - * helicity/polarizations of travelling deformations and mediating quanta
    - * $90°/S$ angle between linear polarization states of classical radiation field of spin S
  - – (Virtual?) field properties
    - * attraction/repulsion (even spin can only attract, odd spin can attract and repel)
  - – Energy spectra and precession
    - * Fine and Hyperfine structure
    - * Thomas precession
- Mathematical Properties
  - * Mathematical spinors, Clifford algebras, Spin(or) representations and groups
  - * Spin-$S \Rightarrow (2S+1)$-dimensional representation of SU(2)
  - * Fiber bundles and spin connections (Fermi-Walker transport?)
  - * 2-spinors versus 4-spinors (How many components can a spinor have?), Dirac, Weyl, Majorana spinors

Approach to Understanding

It would be nice to have a complete understanding of spin, encompassing all of these properties as well as an intuitive understanding of its various applications. Since my intuition is often based on a geometric understanding of concepts that allows me to "see" what's going on, I focus first on the more geometrical concepts to approach complete understanding.

This week I concentrated on these two theorems:

- The classical radiation field of a spin-S particle is always invariant under a rotation of $360°/S$ about its propagation direction.
- A radiation field of any spin S has precisely two orthogonal states of linear polarization inclined to each other at an angle of $90°/S$.

At first I thought it must be that these theorems are assuming the spin S is a positive integer, since the case of spin-half ($S = 1/2$) does not make geometrical sense. (I cannot imagine rotating a classical radiation field, which exists in real space, and not seeing it return to itself after $360°$. I can, however, imagine constructing a toy that measures how much it is rotated, say, by use of an accelerometer, and changes its properties with a period greater than $360°$.) But the theorems do apply to $S = 1/2$ because my source for these theorems, MTW [2] pg. 954, mentions this case (neutrino waves) explicitly. I have a feeling that I am being slightly misled.

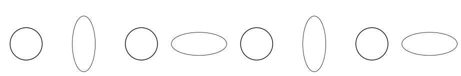
Progress in Understanding

So, what did I learn this week? Actually, the biggest advance was made by reading the paper that Chris found, "What is Spin?" by Hans Ohanian [1], which gives a very physical explanation of spin (described in the list of properties above). More work will have to be done to fully understand the paper, but Professor D'Hoker verified that its contents are "basically what we say" in the more advanced physics classes.

As for the two theorems I concentrated on, Professor D'Hoker gave me a clue for mathematically linking spin-2 polarization modes of deformation to last week's topic of 2×2 symmetric traceless tensors. This link is a part of understanding both theorems for the case of spin 2.

D'Hoker's Clue: Link between Spin-2 Deformations and 2×2 Symmetric Traceless Tensors

I asked how the oscillating circular deformations



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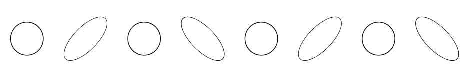


could be related to symmetric traceless 2×2 tensors, and D'Hoker told me that the complex number

$$z = x + iy,$$

when multiplied with itself and its complex conjugate like so,

$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) &= x^2 + y^2 \\ zz &= (x+iy)(x+iy) &= x^2 + 2ixy - y^2 \\ \bar{z}\bar{z} &= (x-iy)(x-iy) &= x^2 - 2ixy - y^2, \end{aligned}$$

give expressions (such as x^2+y^2) that can be used to form circles and ellipses when added in the following fashion:

$$
\begin{aligned}
z\bar{z}+\alpha(zz+\bar{z}\bar{z})+i\beta(zz-\bar{z}\bar{z}) &= (x^2+y^2)+\alpha(2x^2-2y^2)+i\beta(4ixy) \\
&= (1+2\alpha)x^2-4\beta xy+(1-2\alpha)y^2 \\
&= Ax^2+Bxy+Cy^2.
\end{aligned} \tag{1}
$$

So there, equation 1, is a two-parameter expression for circles and ellipses, but how do the tensors relate to this? I basically had to remember myself that I could use a 2×2 matrix as a “quadratic form” to construct the same expression:

$$
\mathbf{x}^T Q\,\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} Ax+By/2 \\ Bx/2+Cy \end{pmatrix} = Ax^2+Bxy+Cy^2,
$$

given that $A=(1+2\alpha)$, $B=-4\beta$, and $A=(1-2\alpha)$. But that means that Q is not traceless (tr $Q=2$):

$$
Q = \begin{pmatrix} 1+2\alpha & -2\beta \\ -2\beta & 1-2\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2\alpha & -2\beta \\ -2\beta & -2\alpha \end{pmatrix}.
$$

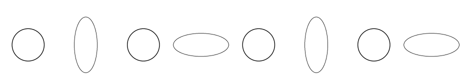
So Q is the addition of *the identity and* a traceless symmetric 2×2 matrix, and since

$$
z\bar{z}+\alpha(zz+\bar{z}\bar{z})+i\beta(zz-\bar{z}\bar{z}) = \mathbf{x}^T Q\,\mathbf{x},
$$

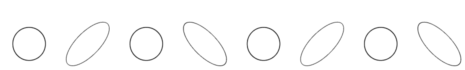
we have the association

$$
z\bar{z}+\alpha(zz+\bar{z}\bar{z})+i\beta(zz-\bar{z}\bar{z}) \;\Leftrightarrow\; Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2\alpha & -2\beta \\ -2\beta & -2\alpha \end{pmatrix}.
$$

Now, I still haven't explicitly shown how the two-parameter expression in equation 1 relates to the two modes of deformations illustrated above. To make the story short, I used the program Mathematica to find out that setting $\{\alpha=\cos t, \beta=0\}$ gives the first mode

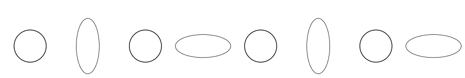
,

and setting $\{\alpha=0, \beta=\cos t\}$ gives the second mode

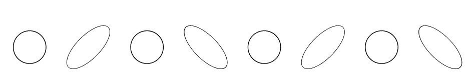
.

(I actually animated a 2-D color-coded contour plot of the oscillating paraboloid deformations.) So the question is, why do those to choices for the parameters $\{\alpha,\beta\}$ yield those shapes? I leave that to the final section of this paper. But we do now have the desired associations:

$$
z\bar{z}+(\cos t)(zz+\bar{z}\bar{z}) \;\Leftrightarrow\; Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\cos t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\Leftrightarrow
$$

,

$$
z\bar{z}+i(\cos t)(zz-\bar{z}\bar{z}) \;\Leftrightarrow\; Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\cos t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\Leftrightarrow
$$

.

Tilted Ellipses and the Parameters $\{\alpha, \beta\}$

Before bringing the parameters α and β into play, there's the question, given an equation for an ellipse, $Ax^2 + Bxy + Cy^2 = D$, at what angle is it tilted? That is, what is the smallest angle between the x-axis and one of the axes of the ellipse? Or, put another way, what is the angle for the (active) coordinate rotation $R(\theta)$ that takes the expression $Ax^2 + Bxy + Cy^2$ in the coordinate system (x, y) to the expression $a(x')^2 + b(y')^2$ in the coordinate system (x', y'), where

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta)\,\mathbf{x} = \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \\ \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathrm{C}\theta\, x - \mathrm{S}\theta\, y \\ \mathrm{S}\theta\, x + \mathrm{C}\theta\, y \end{pmatrix} \ ?$$

The correct angle gives the following relationship:

$$\begin{aligned} a(x')^2 + b(y')^2 &= a(\mathrm{C}\theta\, x - \mathrm{S}\theta\, y)^2 + b(\mathrm{S}\theta\, x + \mathrm{C}\theta\, y)^2 \\ &= a(\mathrm{C}^2\theta\, x^2 - 2\mathrm{C}\theta\mathrm{S}\theta\, xy + \mathrm{S}^2\theta\, y^2) + b(\mathrm{S}^2\theta\, x^2 + 2\mathrm{S}\theta\mathrm{C}\theta\, xy + \mathrm{C}^2\theta\, y^2) \\ &= (a\mathrm{C}^2\theta + b\mathrm{S}^2\theta)x^2 + (b-a)2\mathrm{C}\theta\mathrm{S}\theta\, xy + (a\mathrm{S}^2\theta + b\mathrm{C}^2\theta)y^2 \\ &= Ax^2 + Bxy + Cy^2. \end{aligned}$$

We would like to solve for θ, a, and b in terms of A, B, and C:

$$\begin{aligned} A + C &= (a\mathrm{C}^2\theta + b\mathrm{S}^2\theta) + (a\mathrm{S}^2\theta + b\mathrm{C}^2\theta) \\ &= a + b \\ A - C &= (a\mathrm{C}^2\theta + b\mathrm{S}^2\theta) - (a\mathrm{S}^2\theta + b\mathrm{C}^2\theta) \\ &= (a-b)(\mathrm{C}^2\theta - \mathrm{S}^2\theta) \\ &= (a-b)\mathrm{C}(2\theta) \\ B &= (b-a)2\mathrm{C}\theta\mathrm{S}\theta \\ &= -(a-b)\mathrm{S}(2\theta) \\ \frac{B}{A-C} &= \frac{-(a-b)\mathrm{S}(2\theta)}{(a-b)\mathrm{C}(2\theta)} \\ &= -\mathrm{T}(2\theta) \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} \theta &= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{B}{C-A}\right) \\ a &= \frac{1}{2}(A+C) + \frac{1}{2}\frac{(A-C)}{\mathrm{C}(2\theta)} \\ b &= \frac{1}{2}(A+C) - \frac{1}{2}\frac{(A-C)}{\mathrm{C}(2\theta)} \end{aligned}$$

Now we can plug in our parametrized values for A, B, and C, to solve for θ, a, and b in terms of α and β:

$$\begin{aligned} \theta &= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{B}{C-A}\right) &&= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{-4\beta}{(1-2\alpha)-(1+2\alpha)}\right) &&= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{\beta}{\alpha}\right) \\ a &= \frac{1}{2}(A+C) + \frac{1}{2}\frac{(A-C)}{\mathrm{C}(2\theta)} &&= \frac{1}{2}\big((1+2\alpha)+(1-2\alpha)\big) + \frac{1}{2}\frac{\big((1+2\alpha)-(1-2\alpha)\big)}{\mathrm{C}(2\theta)} &&= 1 + \frac{2\alpha}{\mathrm{C}(2\theta)} \\ b &= \frac{1}{2}(A+C) - \frac{1}{2}\frac{(A-C)}{\mathrm{C}(2\theta)} &&= 1 - \frac{2\alpha}{\mathrm{C}(2\theta)} \end{aligned}$$

Really, though, I care more about the inverses of a and b, since they relate to the semi-major and semi-minor axes:

$$ax^2 + by^2 = \frac{x^2}{R_x} + \frac{y^2}{R_y}.$$

So we have

$$\begin{aligned} \theta &= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{\beta}{\alpha}\right) \\ R_x &= \frac{1}{a} = \frac{1}{1 + \frac{2\alpha}{\mathrm{C}(2\theta)}} \\ R_y &= \frac{1}{b} = \frac{1}{1 - \frac{2\alpha}{\mathrm{C}(2\theta)}} \end{aligned}$$

Well, actually, R_x and R_y only give us the semi-major and semi-minor axes (lengths) if $ax^2 + by^2 = x^2/R_x^2 + y^2/R_y^2 = 1$. What we really care about is the ratio of R_x to R_y, which will tell us the basic shape of any ellipse plotted using the expression $x^2/R_x^2 + y^2/R_y^2$.

$$\begin{aligned} \theta &= \frac{1}{2}\mathrm{T}^{-1}\left(\frac{\beta}{\alpha}\right) \\ \frac{R_x}{R_y} &= \frac{1 - \frac{2\alpha}{\mathrm{C}(2\theta)}}{1 + \frac{2\alpha}{\mathrm{C}(2\theta)}} \end{aligned}$$

I should now show the two modes ($\{\alpha = \cos t, \beta = 0\}$ and $\{\alpha = 0, \beta = \cos t\}$) give what I want, but I have an error for the second mode (I'm dividing by zero).

References

[1] Hans C. Ohanian: "What is spin?" *American Journal of Physics – June 1986 – Volume 54, Issue 6*, pp. 500-505

- This paper may not give the full picture of what spin is, but it seems to provide the most important and physical piece of information about it (and yet I've never seen this description anywhere else!).

[2] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler: *Gravitation*, W. H. Freeman and Company (1973)

- The two basic theorems about spin that I refer to are on page 954.

[3] I. M. Benn, R. W. Tucker: *An Introduction to Spinors and Geometry; with Applications in Physics*, IOP Publishing Ltd (1987)

- Section 2.8 "The confusion of tongues" (pg 85) compares and contrasts the usage of the term spin in mathematics and physics. The book itself covers both perspectives, as it starts with the mathematical perspective, then develops the physics.