2 × 2 Tensor Rotation and Radiation of Spin 2

Classical radiation fields associated with spin-2 mediator particles cause fundamental deformation patterns that, at one moment, are invariant under a rotation of 180°. There is a lot to digest in that statement, but we’ll focus on a simple mathematical theorem that will fit in like a puzzle piece with many other concepts to make the statement understandable. We start with the assertion that spin-2 radiation fields relate (somehow) to symmetric traceless 2-D rank 2 spatial tensors.

**Theorem.** Symmetric traceless 2-D rank 2 spatial tensors have a rotational period of 180°.

**Proof.** A tensor \( T \) that is rank 2 has two indices, and in two dimensions each index runs over two values (1 and 2). We represent such a tensor as \( T_{ij} \). Such a tensor, by definition, transforms under a (2-D, proper) rotation \( R \) to \( T'_{ij} \) in the following manner:

\[
T'_{ij} = R_{ia} R_{jb} T_{ab}
\]

with summation over repeated literal indices, and where the active (as opposed to passive) form of \( R \) is

\[
R_{ij} = R_{ij}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}_{ij} \equiv \begin{pmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{pmatrix}_{ij}.
\]

Usual spatial vectors, on the other hand, transform as

\[
v'_{i} = R_{ia} v_{a}.
\]

The tensor \( T_{ij} \) is assumed to be symmetric (equal to its transpose) and traceless (its trace is equal to zero), so its general form is

\[
T_{ij} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}_{ij}.
\]

With all the assumptions made, we now calculate \( T'_{ij} \) and determine its rotational period.

\[
T'_{11} = R_{1a} R_{1b} T_{ab}
\]

\[
= R_{11} (R_{11} T_{11} + R_{12} T_{12}) + R_{12} (R_{11} T_{21} + R_{12} T_{22})
\]

\[
= C\theta (C\theta a + (-S\theta) b) + (-S\theta) (C\theta b + (-S\theta) (-a))
\]

\[
= a(C^2\theta - S^2\theta) - b(2C\theta S\theta)
\]

\[
= aC(2\theta) - bS(2\theta)
\]

We can write this operation more diagramatically if we define, for any two-indexed object \( M \), a subscripted object \( M_{i} \) that represents a row (rather than a column) of the diagramatic two-by-two-array representation of \( M \):

\[
T'_{11} = R_{1} \left( R_{1} \cdot T_{1} \right) = \begin{pmatrix} C\theta & -S\theta \\ S\theta & C\theta \end{pmatrix} \left( a \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} b \\ -a \end{pmatrix} \right)
\]

\[
= a(C^2\theta - S^2\theta) - b(2C\theta S\theta)
\]

\[
= aC(2\theta) - bS(2\theta)
\]
\[ T'_{12} = R_1 \left( \begin{array}{c} R_2 \cdot T_1 \\ R_2 \cdot T_2 \end{array} \right) = \left( \begin{array}{cc} C\theta & -S\theta \\ S\theta & C\theta \end{array} \right) \left( \begin{array}{c} S\theta & C\theta \\ S\theta & C\theta \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) \]

\[ = \left( \begin{array}{cc} C\theta & -S\theta \\ S\theta & C\theta \end{array} \right) \left( \begin{array}{c} aS\theta + bC\theta \\ bS\theta - aC\theta \end{array} \right) = C\theta(aS\theta + bC\theta) - S\theta(bS\theta - aC\theta) \]

\[ = a(2\cos \theta \sin \theta) + b(C^2 \theta - S^2 \theta) = aS(2\theta) + bC(2\theta) \]

\[ T'_{21} = R_2 \left( \begin{array}{c} R_1 \cdot T_1 \\ R_1 \cdot T_2 \end{array} \right) = \left( \begin{array}{cc} S\theta & C\theta \\ C\theta & -S\theta \end{array} \right) \left( \begin{array}{c} C\theta & -S\theta \\ C\theta & -S\theta \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) \]

\[ = \left( \begin{array}{cc} S\theta & C\theta \\ C\theta & -S\theta \end{array} \right) \left( \begin{array}{c} aC\theta - bS\theta \\ bC\theta + aS\theta \end{array} \right) = S\theta(aC\theta - bS\theta) + C\theta(bC\theta + aS\theta) \]

\[ = a(2\sin \theta \cos \theta) + b(C^2 \theta - S^2 \theta) = aS(2\theta) + bC(2\theta) \]

\[ T'_{22} = R_2 \left( \begin{array}{c} R_2 \cdot T_1 \\ R_2 \cdot T_2 \end{array} \right) = \left( \begin{array}{cc} S\theta & C\theta \\ C\theta & -S\theta \end{array} \right) \left( \begin{array}{c} S\theta & C\theta \\ S\theta & C\theta \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) \]

\[ = \left( \begin{array}{cc} S\theta & C\theta \\ C\theta & -S\theta \end{array} \right) \left( \begin{array}{c} aS\theta + bC\theta \\ bS\theta - aC\theta \end{array} \right) = S\theta(aS\theta + bC\theta) + C\theta(bS\theta - aC\theta) \]

\[ = a(S^2 \theta - C^2 \theta) + b(2S\theta C\theta) = -aC(2\theta) + bS(2\theta) \]

Finally, then, we have

\[ T'_{ij} = \left( \begin{array}{cc} a\cos(2\theta) - b\sin(2\theta) & a\sin(2\theta) + b\cos(2\theta) \\ a\sin(2\theta) + b\cos(2\theta) & -a\cos(2\theta) + b\sin(2\theta) \end{array} \right) \equiv T'_{ij}(\theta), \]

which has a period of \( \pi \), or \( 180^\circ \):

\[ T'_{ij}(\theta + \pi) = T'_{ij}(\theta). \]