Week 1 Lecture: Concepts of Spin

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2×2 Tensor Rotation and Radiation of Spin 2

Classical radiation fields associated with spin-2 mediator particles cause fundamental deformation patterns that, at one moment, are invariant under a rotation of 180°. There is a lot to digest in that statement, but we'll focus on a simple mathematical theorem that will fit in like a puzzle piece with many other concepts to make the statement understandable. We start with the assertion that spin-2 radiation fields relate (somehow) to symmetric traceless 2-D rank 2 spatial tensors.

Theorem. *Symmetric traceless 2-D rank 2 spatial tensors have a rotational period of 180°.*

Proof. A tensor T that is rank 2 has two indices, and in two dimensions each index runs over two values (1 and 2). We represent such a tensor as T_{ij}. Such a tensor, by definition, transforms under a (2-D, proper) rotation R to T'_{ij} in the following manner:

$$T'_{ij} = R_{ia}\, R_{jb}\, T_{ab}$$

with summation over repeated literal indices, and where the active (as opposed to passive) form of R is

$$R_{ij} = R_{ij}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{ij} \equiv \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \\ \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix}_{ij}.$$

Usual spatial vectors, on the other hand, transform as

$$v'_i = R_{ia}\, v_a.$$

The tensor T_{ij} is assumed to be symmetric (equal to its transpose) and traceless (its trace is equal to zero), so its general form is

$$T_{ij} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}_{ij}.$$

With all the assumptions made, we now calculate T'_{ij} and determine its rotational period.

$$\begin{aligned}
T'_{11} &= R_{1a}\, R_{1b}\, T_{ab} \\
&= R_{11}\,(R_{11}\,T_{11} + R_{12}\,T_{12}) + R_{12}\,(R_{11}\,T_{21} + R_{12}\,T_{22}) \\
&= \mathrm{C}\theta\,\big(\mathrm{C}\theta\, a + (-\mathrm{S}\theta)\, b\big) + (-\mathrm{S}\theta)\,\big(\mathrm{C}\theta\, b + (-\mathrm{S}\theta)\,(-a)\big) \\
&= a(\mathrm{C}^2\theta - \mathrm{S}^2\theta) - b(2\mathrm{C}\theta\mathrm{S}\theta) \\
&= a\mathrm{C}(2\theta) - b\mathrm{S}(2\theta)
\end{aligned}$$

We can write this operation more diagramatically if we define, for any two-indexed object M, a subscripted object M_i that represents a row (rather than a column) of the diagramatic two-by-two-array representation of M:

$$\begin{aligned}
T'_{11} &= R_1 \begin{pmatrix} R_1 \cdot T_1 \\ R_1 \cdot T_2 \end{pmatrix} = \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \cdot \begin{pmatrix} a & b \end{pmatrix} \\ \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \cdot \begin{pmatrix} b & -a \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \begin{pmatrix} a\mathrm{C}\theta - b\mathrm{S}\theta \\ b\mathrm{C}\theta + a\mathrm{S}\theta \end{pmatrix} = \mathrm{C}\theta(a\mathrm{C}\theta - b\mathrm{S}\theta) - \mathrm{S}\theta(b\mathrm{C}\theta + a\mathrm{S}\theta) \\
&= a(\mathrm{C}^2\theta - \mathrm{S}^2\theta) - b(2\mathrm{C}\theta\mathrm{S}\theta) = a\mathrm{C}(2\theta) - b\mathrm{S}(2\theta)
\end{aligned}$$

$$
\begin{aligned}
T'_{12} &= R_1 \begin{pmatrix} R_2 \cdot T_1 \\ R_2 \cdot T_2 \end{pmatrix} = \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \cdot \begin{pmatrix} a & b \end{pmatrix} \\ \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \cdot \begin{pmatrix} b & -a \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \begin{pmatrix} a\mathrm{S}\theta + b\mathrm{C}\theta \\ b\mathrm{S}\theta - a\mathrm{C}\theta \end{pmatrix} = \mathrm{C}\theta(a\mathrm{S}\theta + b\mathrm{C}\theta) - \mathrm{S}\theta(b\mathrm{S}\theta - a\mathrm{C}\theta) \\
&= a(2\mathrm{C}\theta\mathrm{S}\theta) + b(\mathrm{C}^2\theta - \mathrm{S}^2\theta) = a\mathrm{S}(2\theta) + b\mathrm{C}(2\theta) \\
T'_{21} &= R_2 \begin{pmatrix} R_1 \cdot T_1 \\ R_1 \cdot T_2 \end{pmatrix} = \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \cdot \begin{pmatrix} a & b \end{pmatrix} \\ \begin{pmatrix} \mathrm{C}\theta & -\mathrm{S}\theta \end{pmatrix} \cdot \begin{pmatrix} b & -a \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \begin{pmatrix} a\mathrm{C}\theta - b\mathrm{S}\theta \\ b\mathrm{C}\theta + a\mathrm{S}\theta \end{pmatrix} = \mathrm{S}\theta(a\mathrm{C}\theta - b\mathrm{S}\theta) + \mathrm{C}\theta(b\mathrm{C}\theta + a\mathrm{S}\theta) \\
&= a(2\mathrm{S}\theta\mathrm{C}\theta) + b(\mathrm{C}^2\theta - \mathrm{S}^2\theta) = a\mathrm{S}(2\theta) + b\mathrm{C}(2\theta) \\
T'_{22} &= R_2 \begin{pmatrix} R_2 \cdot T_1 \\ R_2 \cdot T_2 \end{pmatrix} = \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \cdot \begin{pmatrix} a & b \end{pmatrix} \\ \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \cdot \begin{pmatrix} b & -a \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{S}\theta & \mathrm{C}\theta \end{pmatrix} \begin{pmatrix} a\mathrm{S}\theta + b\mathrm{C}\theta \\ b\mathrm{S}\theta - a\mathrm{C}\theta \end{pmatrix} = \mathrm{S}\theta(a\mathrm{S}\theta + b\mathrm{C}\theta) + \mathrm{C}\theta(b\mathrm{S}\theta - a\mathrm{C}\theta) \\
&= a(\mathrm{S}^2\theta - \mathrm{C}^2\theta) + b(2\mathrm{S}\theta\mathrm{C}\theta) = -a\mathrm{C}(2\theta) + b\mathrm{S}(2\theta)
\end{aligned}
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Finally, then, we have

$$
T'_{ij} = \begin{pmatrix} a\mathrm{C}(2\theta) - b\mathrm{S}(2\theta) & a\mathrm{S}(2\theta) + b\mathrm{C}(2\theta) \\ a\mathrm{S}(2\theta) + b\mathrm{C}(2\theta) & -a\mathrm{C}(2\theta) + b\mathrm{S}(2\theta) \end{pmatrix}_{ij} \equiv T'_{ij}(\theta),
$$

which has a period of π, or $180°$:

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T'_{ij}(\theta + \pi) = T'_{ij}(\theta).
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