Week 1 Lecture: Concepts of Quantum Field Theory (QFT)

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Relative Wave-Functional Probabilities

This Week's Questions

- What are the exact solutions for the Klein-Gordon field?
  - What about for two Klein-Gordon-coupled oscillators?
  - What about for a lattice of Klein-Gordon-coupled oscillators?
- Are the energy eigen-QFT-states also configuration eigenstates of the field?
  - Might the energy eigen-QFT-states actually be modes of the collective wavefunctions, rather than modes of the collective (point-like) oscillators? This would mean that for a given energy there is only a high probability that the field takes on a planewave configuration.
  - Does the Hamiltonian (energy) operator $H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, \omega_{\mathbf{p}} \left(a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] \right)$ act on fields or state-functions?
- What is the intuitive version of $\Psi_0[\tilde{\varphi}]$?
- Is the wave-functional ratio $\Psi_0[\varphi]/\Psi_0[0]$ the relative probability (density) that the ground state field will be measured as (or will interact as) φ?
  - According to Kanatchikov [2], it is.

Important Information Gained from the Literature

Kanatchikov

An important claim is made by Kanatchikov [2] (in the first paragraph of section 4): "…the Schrödinger wave functional $\Psi_{\mathrm{S}}\big([y^a(\mathbf{x})], t\big)$ …is known to be the probability amplitude of observing the field *configuration* $y^a(\mathbf{x})$ on a space-like hypersurface of constant time t."

Kanatchikov also proposes the intuitive wave-functional formula that I propose! (See comments in the References.)

Jackiw

Some important comments are made by Jackiw [4] (shown in the References below), along with the following. (I've condensed and slightly changed the material to suit my notational conventions.)

States are viewed as functionals $\Psi[\varphi]$ of a c-number field at fixed time $\varphi(x)$. An inner product is defined by functional integration.

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}\varphi \; \Psi_1^*[\varphi] \, \Psi_2[\varphi] = \langle \Psi_2 | \Psi_1 \rangle^* \,.$$

The operators $\Phi(x)$ and $\Pi(x)$ act on states as follows

$$\begin{aligned} \Phi(x)\,\Psi[\varphi] &= \varphi(x)\,\Psi[\varphi], \\ \Pi(x)\,\Psi[\varphi] &= -i\tfrac{\delta}{\delta\varphi(x)}\,\Psi[\varphi]. \end{aligned}$$

"While the analogy with quantum mechanics is obvious, a difference emerges when we consider a Fock basis for our functional space. This basis is the field theoretic analog of a harmonic oscillator basis in quantum mechanics. A *Fock vacuum* is a Gaussian functional with covariance Ω, which is symmetric [and] can be complex, but possesses a positive definite real part

$$\Psi_\Omega[\varphi] = \det{}^{1/4}\!\left(\frac{\Omega_{\mathrm{r}}}{\pi}\right) \exp\left[-\frac{1}{2}\int \varphi\Omega\varphi\right],$$

$$\Omega = \Omega_{\mathrm{r}} + i\Omega_{\mathrm{i}}, \qquad \Omega(x,y) = \Omega(y,x).$$

An obvious functional notation is used throughout[1],

$$\int \varphi\Omega\varphi \equiv \int \mathrm{d}x\,\mathrm{d}y \;\varphi(x)\,\Omega(x,y)\,\varphi(y),$$

and the determinant is functional. In spite of the possibility for greater generality, we shall always use translational invariant covariances,

$$\Omega(x,y) = \int \frac{\mathrm{d}p}{2\pi} e^{-ip(x-y)}\,\Omega(p),$$

which are diagonal in the momentum representation.

$$\int \mathrm{d}x\,\mathrm{d}y\; e^{ipx}\,\Omega(x,y)\,e^{-iqy} = \Omega(p)\,(2\pi)\,\delta(p-q).$$

Higher basis states are polynomials in φ multiplying Ψ_Ω, and they are orthonormalized if linear combinations corresponding to 'functional Hermite polynomials' are taken. This defines a *Fock space* within our functional space.

[1]Note: This looks close to the integral I obtained in the Bonus Lecture from last quarter (in "Attempt 2") – $\int \mathrm{d}^3x_1\,\mathrm{d}^3x_2\;\varphi(\mathbf{x}_1)\,\varphi(\mathbf{x}_2)$.

In our functional space, different Fock bases constructed with different covariances can be inequivalent. This happens because field theory possesses an infinite number of degrees of freedom. For consider two Fock vacua with covariances Ω_1 and Ω_2. Their overlap is

$$e^{-N} \equiv \langle \Omega_1 | \Omega_2 \rangle \,.$$

For example when the covariances are real, N is given by

$$N = \frac{L}{2} \int \frac{\mathrm{d}k}{2\pi} \;\ln \frac{1}{2} \left(\sqrt{\frac{\Omega_1(k)}{\Omega_2(k)}} + \sqrt{\frac{\Omega_2(k)}{\Omega_1(k)}} \right) ,$$

where L is the length of the space."

Note: I'm not sure what equation (2.7) means.

The Intuitive Wave-Functional of the Fourier-Transformed Field

The state-function (in terms of a partial Fourier transform) and a partial Fourier transform (in terms of the usual state-function) are

$$\Psi = \Psi(\mathbf{x}, \phi) = \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \; \Psi(\mathbf{p}, \phi)\, e^{i\mathbf{p}\cdot\mathbf{x}} \qquad\qquad \Psi(\mathbf{p}, \phi) = \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \Psi(\mathbf{x}, \phi)\, e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

The full Fourier transform of the state-function is

$$\tilde{\Psi} = \tilde{\Psi}(\mathbf{p}, \pi) = \Psi(\mathbf{p}, \pi).$$

The field and its Fourier transform are

$$\varphi = \varphi(\mathbf{x}) = \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \varphi(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \qquad\qquad \tilde{\varphi} = \tilde{\varphi}(\mathbf{p}) = \varphi(\mathbf{p}) = \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \varphi(\mathbf{x})\, e^{-i\mathbf{p}\cdot\mathbf{x}}.$$

So, let's try to express the wave-functional

$$\Psi[\varphi] = \prod_{\mathbf{x}} \Psi\big(\mathbf{x}, \varphi(\mathbf{x})\big) = \prod_{\mathbf{x}} e^{\ln \Psi(\mathbf{x}, \varphi(\mathbf{x}))} = e^{\sum_{\mathbf{x}} \ln \Psi(\mathbf{x}, \varphi(\mathbf{x}))} = e^{\int \mathrm{d}^3 x \, \ln \Psi(\mathbf{x}, \varphi(\mathbf{x}))}$$

in terms of the Fourier-transformed field:

$$\begin{aligned}
\Psi[\tilde{\varphi}] &= \prod_{\mathbf{x}} \Psi\big(\mathbf{x}, \varphi(\mathbf{x})\big) \\
&= \prod_{\mathbf{x}} \Psi\left(\mathbf{x}, \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \tilde{\varphi}(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \\
&= \prod_{\mathbf{x}} \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \; \Psi\left(\mathbf{p}, \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \tilde{\varphi}(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \; e^{i\mathbf{p}\cdot\mathbf{x}} \\
&= \exp\left[\int \mathrm{d}^3 x \;\ln \left\{ \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \; \Psi\left(\mathbf{p}, \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \; \tilde{\varphi}(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \; e^{i\mathbf{p}\cdot\mathbf{x}} \right\} \right] \\
&= \prod_{\mathbf{p}} ?
\end{aligned}$$

It'd be nice if it were something like $\prod_{\mathbf{p}} \Psi\big(\mathbf{p}, \tilde{\varphi}(\mathbf{p})\big)$

Graphing the Wave-Functional on the Field Fourier Transform

I solved this initially last quarter in my hand-written notes for Lecture 9 of Winter 2008.

The wave-functional $\Psi_0[\tilde{\varphi}]$ on the field Fourier transform $\tilde{\varphi}$ is given in Hatfield [1] Eqn. (10.26) as

$$\Psi_0[\tilde{\varphi}] \quad \propto \quad \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\sqrt{\mathbf{k}^2+m^2}\,\tilde{\varphi}(\mathbf{k})\,\tilde{\varphi}(-\mathbf{k})\right]$$

using $\omega_k = \sqrt{\mathbf{k}^2+m^2}$, given at the top of that page (pg 203). Let's assume that the wave-functional ratio $\Psi_0[\tilde{\varphi}]/\Psi_0[0]$ is the relative probability (density) that the ground state (Fourier transformed) field will be measured as (or will interact as) $\tilde{\varphi}$.

- Q: How likely is it that the ground state manifests itself as a planewave eigen-field $\varphi_{\mathbf{q}}$?

  A: Just as likely as it manifests itself as being completely flat. (Except for a "planewave" that is 1 everywhere.)

$$\varphi_{\mathbf{q}}(\mathbf{x}) = \mathrm{Re}\left(e^{i\mathbf{q}\cdot\mathbf{x}}\right) = \cos(\mathbf{q}\cdot\mathbf{x})$$

$$\underset{\sim}{\varphi}_{\mathbf{q}}(\mathbf{x}) = e^{i\mathbf{q}\cdot\mathbf{x}}$$

$$\begin{aligned}\tilde{\underset{\sim}{\varphi}}_{\mathbf{q}}(\mathbf{p}) &= \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\;\underset{\sim}{\varphi}_{\mathbf{q}}(\mathbf{x})\,e^{-i\mathbf{p}\cdot\mathbf{x}} \\ &= \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\;e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \\ &= (2\pi)^{3/2}\,\delta(\mathbf{q}-\mathbf{p})\end{aligned}$$

$$\begin{aligned}\Psi_0[\tilde{\varphi}_{\mathbf{q}}]/\Psi_0[0] &= \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\sqrt{\mathbf{k}^2+m^2}\,(2\pi)^{3/2}\,\delta(\mathbf{q}-\mathbf{k})\,(2\pi)^{3/2}\,\delta(\mathbf{q}+\mathbf{k})\right] \\ &= \exp\left[-\frac{1}{2}\int \mathrm{d}^3 k\;\sqrt{\mathbf{k}^2+m^2}\,\delta(\mathbf{q}-\mathbf{k})\,\delta(\mathbf{q}+\mathbf{k})\right] \\ &= \exp\left[-\frac{1}{2}\sqrt{\mathbf{q}^2+m^2}\,\delta(2\mathbf{q})\right] \\ &= \begin{cases} e^0 = 1 & \mathbf{q}\neq\mathbf{0} \\ \text{impossible} & \mathbf{q}=\mathbf{0}\end{cases}\end{aligned}$$

- Q: How likely is it that the ground state manifests itself as a Gaussian eigen-field φ_{G}?

  A: Some Gaussians are equally likely, some are less likely, some are more (even infinitely more) likely.

$$\varphi_{\mathrm{G}}(\mathbf{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 e^{-\mathbf{x}^2/2\sigma^2}$$

$$\varphi_{\mathrm{G}}(\mathbf{x}) = \left(\frac{a}{\pi}\right)^{3/2} e^{-a\mathbf{x}^2} \qquad a = 1/2\sigma^2$$

$$
\begin{aligned}
\tilde{\varphi}_{\mathrm{G}}(\mathbf{p}) &= \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\ \varphi_{\mathrm{G}}(\mathbf{x})\, e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \left(\frac{a}{\pi}\right)^{3/2} \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\ e^{-a\mathbf{x}^2-i\mathbf{p}\cdot\mathbf{x}} \\
&= \left(\frac{a}{\pi}\right)^{3/2} \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\ e^{-a(\mathbf{x}^2+i\mathbf{p}\cdot\mathbf{x}/a)} \\
&= \left(\frac{a}{\pi}\right)^{3/2} \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\ e^{-a(\mathbf{x}+i\mathbf{p}/2a)^2}\ e^{-\mathbf{p}^2/4a} \\
&= \left(\frac{a}{\pi}\right)^{3/2}\ e^{-\mathbf{p}^2/4a}\ \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3u\ e^{-a\mathbf{u}^2} \\
&= \left(\frac{a}{\pi}\right)^{3/2}\ e^{-\mathbf{p}^2/4a}\ \frac{1}{(2\pi)^{3/2}}\ \left(\frac{\pi}{a}\right)^{3/2} \\
&= \frac{1}{(2\pi)^{3/2}}\ e^{-\mathbf{p}^2/4a} \\
&= (2\pi)^{-3/2}\ e^{-\sigma^2\mathbf{p}^2/2}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_0[\tilde{\varphi}_{\mathrm{G}}]/\Psi_0[0] &= \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\mathbf{k}^2+m^2}\,(2\pi)^{-3/2}\,e^{-\sigma^2\mathbf{k}^2/2}\,(2\pi)^{-3/2}\,e^{-\sigma^2(-\mathbf{k})^2/2}\right] \\
&= \exp\left[-\frac{1}{2}\int \mathrm{d}^3k\sqrt{\mathbf{k}^2+m^2}\,e^{-\sigma^2\mathbf{k}^2}\right]
\end{aligned}
$$

Mathematica doesn't know how to solve this 3-D version, so let's go to 1-D:

$$
\begin{aligned}
\Psi_0[\tilde{\varphi}_{\mathrm{G}}]/\Psi_0[0] &= \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}k}{(2\pi)}\sqrt{k^2+m^2}\,(2\pi)^{-1/2}\,e^{-\sigma^2k^2/2}\,(2\pi)^{-1/2}\,e^{-\sigma^2(-k)^2/2}\right] \\
&= \exp\left[-\frac{1}{2}\int \mathrm{d}k\sqrt{k^2+m^2}\,e^{-\sigma^2k^2}\right] \\
&= \exp\left[-\frac{\sqrt{\pi}}{2\sigma^2}\ \mathrm{HypergeometricU}\left[-\frac{1}{2},0,\sigma^2m^2\right]\right]
\end{aligned}
$$

I solved and graphed this using Mathematica. There are many points (σ, m) where this expression is greater than 1 (and approaches infinity) and where it's less than 1.

- Comment: This does not follow my intuition. I would expect the flat ($\varphi = 0$) field to be the most likely configuration. Planewaves and Gaussians should not be equally likely and should not be more likely.

References

[1] Brian Hatfield: *Quantum Field Theory of Point Particles and Strings*, Addison Wesley Longman, Inc. (1992)

- Chapter 10: Free Fields in the Schrödinger Representation

[2] Igor V. Kanatchikov: *Precanonical quantization and the Schrdinger wave functional*, Physics Letters A Volume 283, Issues 1-2, 7 May 2001, Pages 25-36

- This is the first paper that I've found that contains the intuitive formula for the wave-functional. In Kanatchikov's notation (where Σ is a Cauchy surface, etc.):

$$\Psi\big([y^a(\mathbf{x})],t\big) = \prod_{\mathbf{x}\in\Sigma} \Psi|_\Sigma = \prod_{\mathbf{x}} \Psi\big(y^a(\mathbf{x}),\mathbf{x},t\big)$$

$$\prod_{\mathbf{x}} \Psi\big(y^a(\mathbf{x}),\mathbf{x},t\big) = e^{\sum_{\mathbf{x}} \ln \Psi\left(y^a(\mathbf{x}),\mathbf{x},t\right)}$$

  Additional statements are made that I have not made or verified, such as

$$e^{\sum_{\mathbf{x}} \ln \Psi\left(y^a(\mathbf{x}),\mathbf{x},t\right)} = \lim_{\Delta x\to 0} e^{(1/(\Delta x)^3)\int \mathrm{d}\mathbf{x} \ln \Psi\left(y^a(\mathbf{x}),\mathbf{x},t\right)}$$

- Also, an important claim is made (in the first paragraph of section 4):

  "...the Schrödinger wave functional $\Psi_{\mathrm{S}}\big([y^a(\mathbf{x})],t\big)$ ...is known to be the probability amplitude of observing the field *configuration* $y^a(\mathbf{x})$ on a space-like hypersurface of constant time t."

  This is what I've been assuming in my calculations in this lecture and prior lectures. (And this is an answer to one of the questions I asked this week.)

[3] jostpuur: *wave functional* (Physics Forum Thread) < `http://www.physicsforums.com/archive/index.php/t-180190.html` > (Accessed 03 April 2008)

- This person seems to have questions similar to ours.

[4] R. Jackiw: *Schrödinger Picture for Boson and Fermion Quantum Field Theories*, Canadian Mathematical Society (CMS) Conference Proceedings, Volume 9; Mathematical Quantum Field Theory and Related Topics: Proceedings of the 1987 Montréal Conference held September 1-5, 1987; Published by the American Mathematical Society (1988)

- I forgot to list this book in a previous lecture, but this is only the second source I've found that introduces the Schrödinger picture in any depth.
- Important: "The Schrödinger picture approach gains a *unique advantage* when analyzing time-dependent problems, like field theory in de Sitter space or out of thermal equilibrium, where the concept of Fock vacuum is ill-defined." It also has a "*stylistic advantage*" over the conventional approach "because one can well-define renormalized generators without normal ordering with respect to a pre-selected vacuum."