Notes for Physics 6B

Three Related Right Hand Rules

0 Three Rules, Many Styles

There seem to be three right hand rules (RHRs):

- Cross Product RHR
- Loop RHR
- Grip RHR

There are many styles for applying the first two rules, by which I mean that there many ways to use your right hand that yield the same result. My favorite style, which I think is superior to the others, is what I call the Monkey style.

1 Cross Product RHR

Let's perform the right hand rule for the vector cross product $\mathbf{a} \times \mathbf{b}$. First, whether or not these vectors $\mathbf{a}$ and $\mathbf{b}$ are located near each other, imagine translating them so that they originate from the same point, with their "tails" touching. Preserve their directions when translating.

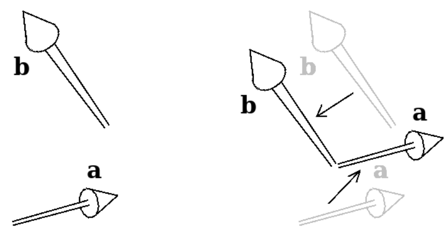

Figure 1: (Mentally) Arrange the Vectors

Now cause your hand to take a flat planar form, with your fingers straight and touching side-by-side and your thumb out at a 90° angle from your fingers. Stick your flat hand in the direction of the first vector ($\mathbf{a}$), leaving your hand free to rotate around the axis of your forearm. Now rotate your flattened hand such that your fingers can bend toward the second vector ($\mathbf{b}$). The direction of the cross product vector ($\mathbf{c} = \mathbf{a} \times \mathbf{b}$) is given by your thumb.

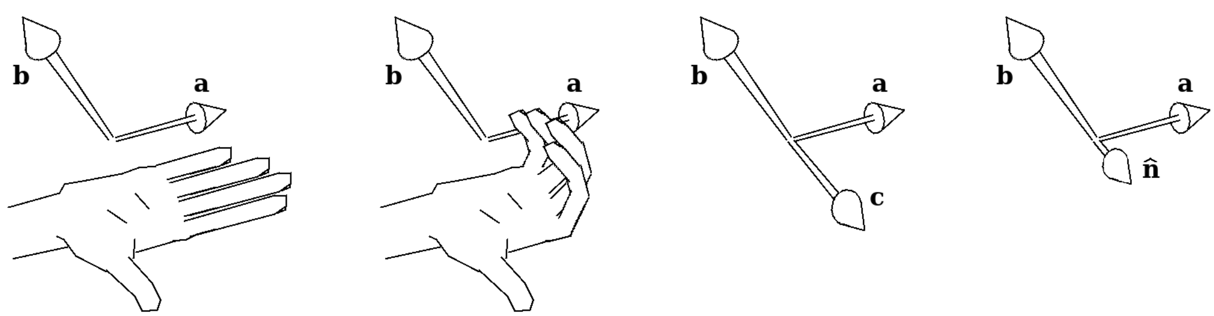

Figure 2: Cross Product Right Hand Rule, Monkey Style

Note that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and perpendicular to $\mathbf{b}$ and thus it is also perpendicular to the plane that contains $\mathbf{a}$ and $\mathbf{b}$. We refer to the unit vector $\hat{\mathbf{n}}$ in Section 4.

Examples

Here are some examples of uses of the Cross Product RHR:

- Relationship between axes the usual right-handed coordinate system: $\hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$
- Biot-Savart law ($\mathbf{B}$ due to a current distribution): $\mathbf{B} = \int K_\mathrm{m} \frac{I \mathrm{d}\mathbf{s} \times \hat{\mathbf{r}}}{r^2}$
- Magnetic force on a point charge: $\mathbf{F}_\mathrm{m} = q\mathbf{v} \times \mathbf{B}$

  When q is negative, the force is in the opposite direction of $\mathbf{v} \times \mathbf{B}$.

  (For negative charges, such as electrons, you can use a left-hand rule to find the direction of the force.)
- Magnetic force on a current-carrying segment of wire: $\mathbf{F}_\mathrm{m} = I\boldsymbol{\ell} \times \mathbf{B}$

  When I is negative, the force is in the opposite direction of $\boldsymbol{\ell} \times \mathbf{B}$.

2 Loop RHR

The Loop RHR or Loop/Orientation RHR, relates loop direction with threading of flows through the loop (as in Ampère's law), and it relates loop direction with surface area vectors (as in Faraday's law).

When using Ampère's law,

$$\oint_P \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \mu_0 I_\mathrm{enc},$$

you draw an imaginary closed loop[1] P and decide which direction you plan to orient your line integral of $\mathbf{B}$. Once you choose that direction, you use this right hand rule to determine the signs you will assign to any currents that are encircled by this loop. Then you can properly add the currents that contribute to the total encircled current I_enc. Any currents through the loop are said to "thread" the loop.

Put your hand along the loop in the direction you have decided to integrate. Your thumb gives the direction for positive contributions. So, referring to the figure below, we have $I_\mathrm{enc} = I_1 - I_2 + I_3$.

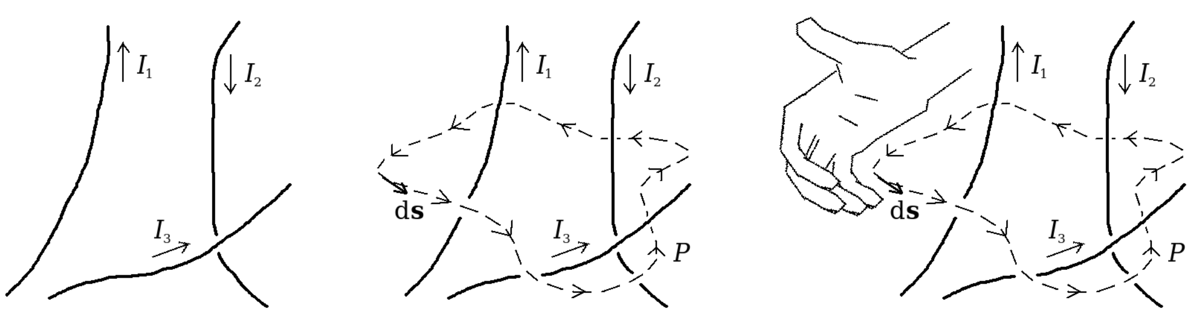

Figure 3: Loop/Threading Right Hand Rule, Monkey Style

[1] You can call this loop an Ampèrian loop, just like we called our imaginary closed surfaces Gaussian surfaces when using Gauss's Law.

When using Faraday's law of induction,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t},$$

which, in more detail, is

$$\oint_P \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

you draw an imaginary closed loop[2] P, which usually coincides with a wire (since you are interested in an induced current in the wire), and decide which direction you plan to orient your arrow for the induced emf $\mathcal{E}$. (The emf arrow goes in the same direction you perform the line integral of $\mathbf{E}$.) Once you choose that direction, you must also choose a surface S that is bounded by the loop P. The surface can be any surface whatsoever, so long as it is bounded by P; S can be flat (if P exists in a plane) or it can bow outward or bend back and forth. Usually, we pick S to be flat, if possible. With P, S, and a direction along P chosen, we use this right hand rule to determine the direction of the infinitesimal area vectors d$\mathbf{a}$ in the surface integral of $\mathbf{B}$. This will determine the "orientation" of the surface S and the sign of the magnetic flux Φ_B.

Put your hand along the loop in the direction of the emf arrow. Your thumb gives the side of the surface on which the infinitesimal surface area vectors d$\mathbf{a}$ go.

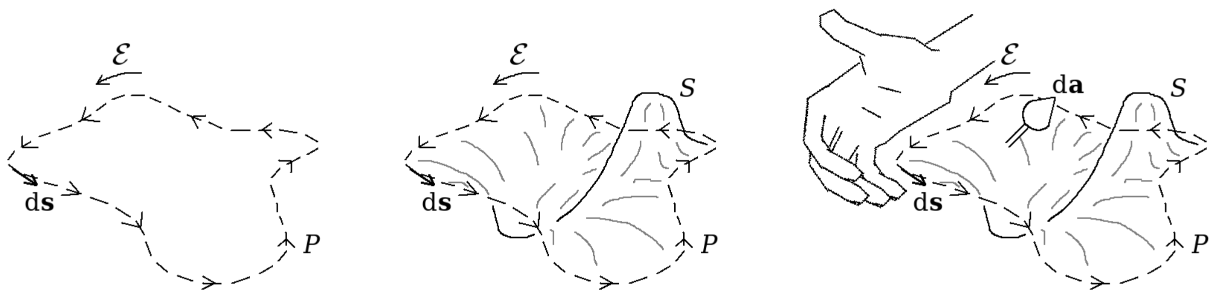

Figure 4: Loop/Surface Right Hand Rule, Monkey Style

Examples

Here are some examples of uses of the Loop RHR:

- Ampère's law – Loop/Threading
- Faraday's law (of induction) – Loop/Surface
- Ampère-Maxwell law – Loop/Threading and Loop/Surface
- Kelvin-Stokes Curl theorem – Loop/Surface

[2]You can call this loop a Faradayan loop, similar to the Ampèrian loop and Gaussian surface.

3 Grip RHR

The Grip RHR or Current/Circulation RHR relates the direction of the current in a wire to the direction of circulation of the magnetic field around the wire.

Imagine grabbing the wire so that your thumb points in the direction of the current arrow for I. Your fingers wrap around the wire and provide a reference direction $\hat{\boldsymbol{\phi}}$ for circulation around the wire. If the current I is positive, then the magnetic field $\mathbf{B}$ circulates in the reference direction. If the current I is negative, then $\mathbf{B}$ circulates in the opposite direction.

For an infinitely long wire with current I, the magnetic field $\mathbf{B}$ due to the current is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}},$$

where r (the magnitude of $\mathbf{r}$) is the perpendicular distance from the wire to the location of $\mathbf{B}$.

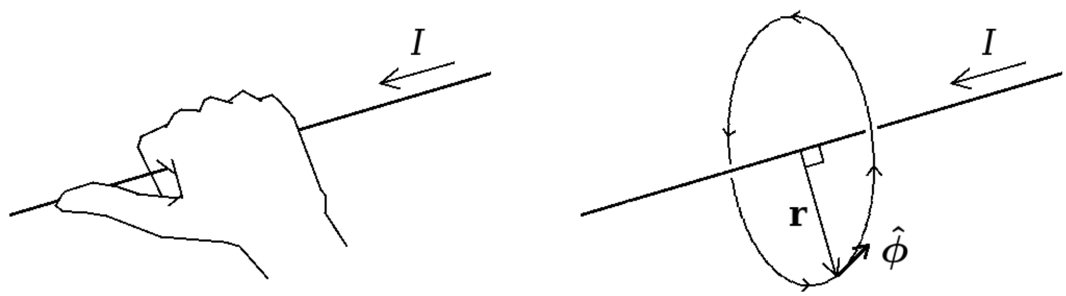

Figure 5: Grip Right Hand Rule

In addition to figuring the $\mathbf{B}$-circulation direction for a straight wire, you can use the grip RHR to help figure out which direction the magnetic field points in more complicated situations. For example, the grip RHR helps in deciding which direction the magnetic field points inside a solenoid.

4 Further Notes

Geometrically, we define the (vector-valued) cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ with the following formula,

$$\mathbf{a} \times \mathbf{b} \equiv |\mathbf{a}|\,|\mathbf{b}| \sin\theta_{ab} \hat{\mathbf{n}} = ab \sin\theta_{ab} \hat{\mathbf{n}},$$

where we take θ_{ab} to be the smallest angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$, and the unit vector direction $\hat{\mathbf{n}}$ is given by the cross-product right-hand-rule (see Figure 2). In the plane of $\mathbf{a}$ and $\mathbf{b}$, there are two angles between them; we pick the smaller one to be θ_{ab}. Thus θ_{ab} is between 0 and π radians, so $\sin\theta_{ab}$ is positive.

There is an amazing little formula that uses the 3-dimensional determinant that allows us to calculate explicitly what the cross product of two vectors ($\mathbf{a}\times\mathbf{b}$) is and also, thus, calculate explicitly what the direction of the cross product ($\hat{\mathbf{n}}$) is.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \left\langle \begin{array}{c} a_x \\ a_y \\ a_z \end{array} \right\rangle \times \left\langle \begin{array}{c} b_x \\ b_y \\ b_z \end{array} \right\rangle \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= (a_y b_z - a_z b_y)\hat{\mathbf{x}} - (a_x b_z - a_z b_x)\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\hat{\mathbf{z}} \\
&= \left\langle \begin{array}{c} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{array} \right\rangle
\end{aligned}$$

Given that this is true, we have

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta_{ab} = \sqrt{(a_y b_z - a_z b_y)^2 + (a_x b_z - a_z b_x)^2 + (a_x b_y - a_y b_x)^2}$$

and so we have

$$\begin{aligned}
a &= \sqrt{a_x^2 + a_y^2 + a_z^2} \\
b &= \sqrt{b_x^2 + b_y^2 + b_z^2} \\
\sin\theta_{ab} &= \frac{|\mathbf{a} \times \mathbf{b}|}{ab} = \frac{\sqrt{(a_y b_z - a_z b_y)^2 + (a_x b_z - a_z b_x)^2 + (a_x b_y - a_y b_x)^2}}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \\
\hat{\mathbf{n}} &= \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{(a_y b_z - a_z b_y)\hat{\mathbf{x}} - (a_x b_z - a_z b_x)\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\hat{\mathbf{z}}}{\sqrt{(a_y b_z - a_z b_y)^2 + (a_x b_z - a_z b_x)^2 + (a_x b_y - a_y b_x)^2}}
\end{aligned}$$

References

[1] Raymond A. Serway; John W. Jewett, Jr.: *Principles of Physics Vol. 2: 6B / 6C - UCLA, Fourth Edition*, Cengage Learning (2008)