Physics 1C: Problem 31 from Chapter 28 of Giancoli

The Magnetic Field in the Center of a Current Distribution

31. A circular conducting ring of radius R is connected to two exterior straight wires ending at two ends of a diameter (Fig. 1). The current I splits into unequal portions while passing through the ring as shown. What is $\mathbf{B}$ at the center of the ring?

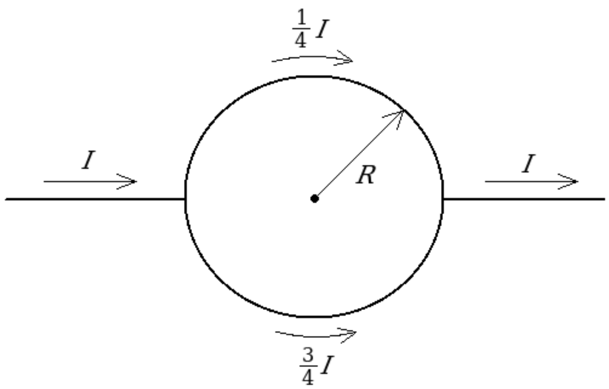

Figure 1: The conducting ring of wire.

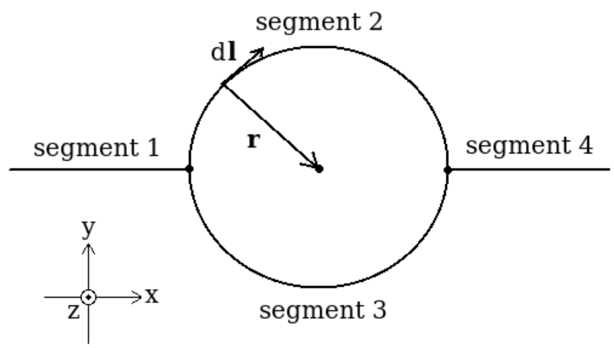

Figure 2: Picking a coordinate system, labels, and a representative "source" point.

Initial observations

As with any problem, it is good to get an idea of what is physically occuring and see if you can qualitatively reason out what the answer should be. (I'll explain these observations in more detail below.) We can see that there will be no contribution to $\mathbf{B}$ from segment 1 (Fig. 2) because (using the Biot-Savart law) the vectors $\mathrm{d}\mathbf{l}$ and $\hat{\mathbf{r}}$ for that segment are parallel, so $\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}} = \mathbf{0}$. There is also no contribution from segment 4 since $\mathrm{d}\mathbf{l}$ and $\hat{\mathbf{r}}$ are antiparallel, or in opposite directions, so $\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}} = \mathbf{0}$ again. Thus only the currents from segments 2 and 3 act as sources for $\mathbf{B}$.

Also, we see by using the right-hand rule that segment 2 will contribute magnetic field pointing into-the-paper while segment 3 will contribute out-of-the-paper (assuming I is positive). However, since the current in segment 3 is greater in magnitude, it will win out, so $\mathbf{B}$ will point out-of-the-paper (if I is positive, or into-the-paper if I is negative).

The calculation

I'll write out the calculation in one long string, labelling each equation so as to explain each step below. Note that before beginning a calculation you should try to define as many variables as you know you'll need, often using diagrams, and during a calculation you should define or redefine what you need as the need arises.

At the center of the ring, we have

$$\mathbf{B} = \int \mathrm{d}\mathbf{B} \qquad (1)$$

$$= K_\mathrm{m} \int \frac{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \tag{2}$$

$$= K_\mathrm{m} \left[\int\limits_{\text{segment 1}} \frac{\cancelto{\mathbf{0}}{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}}{r^2} + \int\limits_{\text{segment 2}} \frac{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2} + \int\limits_{\text{segment 3}} \frac{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2} + \int\limits_{\text{segment 4}} \frac{\cancelto{\mathbf{0}}{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}}{r^2} \right] \tag{3}$$

$$= K_\mathrm{m} \left[\int\limits_{\text{seg. 2}} \frac{\left(\frac{1}{4}I\right)(-\mathrm{d}l\,\hat{\mathbf{z}})}{(R)^2} + \int\limits_{\text{seg. 3}} \frac{\left(\frac{3}{4}I\right)(\mathrm{d}l\,\hat{\mathbf{z}})}{(R)^2} \right] \tag{4}$$

$$= K_\mathrm{m} \frac{I}{4R^2}\,\hat{\mathbf{z}} \left[-\int\limits_{\text{seg. 2}} \mathrm{d}l + 3 \int\limits_{\text{seg. 3}} \mathrm{d}l \right] \tag{5}$$

$$= K_\mathrm{m} \frac{I}{4R^2}\,\hat{\mathbf{z}} \left[-(\pi R) + 3(\pi R) \right] \tag{6}$$

$$= \left(\frac{\mu_0}{4\pi}\right) \frac{I}{4R^2}\,\hat{\mathbf{z}}\,(2\pi R) \tag{7}$$

$$= \frac{\mu_0 I}{8R}\,\hat{\mathbf{z}} \tag{8}$$

Explanation of each step

(1) Magnetic fields originate from two things. One is the movement of electric charges, which can be in the form of current in a wire or electrons whizzing around in atoms. The other is something called spin, which we won't examine in this class. (The magnetic field of a magnet is due to both to the electrons' orbits and the electrons' spins.) In this problem, the only source is current in wires. So the magnetic field originates from many infinitesimal sources: each piece of the wires that has current can be seen as a source for the field. We have to add, or integrate, over the contributions $\mathrm{d}\mathbf{B}$ from each of these infinitesimal sources. (The range of the integration is implied to be "over all possible sources".)

(2) We use the Biot-Savart law, in its traditional form that refers to current (rather than point charges): the infinitesimal magnetic field $\mathrm{d}\mathbf{B}$ at a position $\mathbf{x}$ due to a current i (with its direction, or arrow, defined in the direction $\hat{\mathrm{d}\mathbf{l}}$) over an infinitesimal distance $\mathrm{d}l$ located at position $\mathbf{x}'$ is

$$\mathrm{d}\mathbf{B} = K_\mathrm{m} \frac{i\,\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2},$$

where $K_\mathrm{m} = \mu_0/4\pi$ is the magnetostatic constant, analogous to the electrostatic constant (Coulomb's constant) $K_\mathrm{e} = 1/4\pi\varepsilon_0$, and $\mathbf{r} = r\hat{\mathbf{r}} \equiv \mathbf{x} - \mathbf{x}'$.

Usually, I use the symbols I and $\mathbf{R}$ for the variables in the Biot-Savart law, but since this problem declairs I and R to be constants, I switched to using i and $\mathbf{r}$.

Note that $\mathbf{r}$ goes from the "source point" $\mathbf{x}'$ to the point of interest $\mathbf{x}$, which for us is the center of the ring.

Note also that when using this equation you should point the infinitesimal vector $\mathrm{d}\mathbf{l}$ and the current arrow for i in the same direction.

(3) The integral is broken into four pieces that correspond to the segments in Figure 2. For segment 1, $\mathrm{d}\mathbf{l}$ points rightward and $\hat{\mathbf{r}}$ points rightward, so the angle between them is 0° and $|\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}| = (\mathrm{d}l)(1)\sin(0°) = (\mathrm{d}l)(0) = 0$. For segment 4, $\mathrm{d}\mathbf{l}$ points rightward and $\hat{\mathbf{r}}$ points leftward, so the angle between them is 180° and $|\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}| = (\mathrm{d}l)(1)\sin(180°) = (\mathrm{d}l)(0) = 0$. So the first and last integrals are zero.

(4) We now substitute the appropriate values for the variables in each integral. For segment 2, we have $i = \frac{1}{4}I$, $r = R$, $\hat{\mathbf{r}}$ remains variable, and

$$\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}} = \left|\mathrm{d}l\,\hat{\mathrm{d}\mathbf{l}}\right| |\hat{\mathbf{r}}| \sin(90°) \left(\hat{\mathrm{d}\mathbf{l}} \times \hat{\mathbf{r}}\right) = (\mathrm{d}l)(1)(1)(-\hat{\mathbf{z}}) = -\mathrm{d}l\,\hat{\mathbf{z}}.$$

For segment 3, we have $i = \frac{3}{4}I$, $r = R$, $\hat{\mathbf{r}}$ remains variable, and

$$\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}} = \left|\mathrm{d}l\,\hat{\mathrm{d}\mathbf{l}}\right| |\hat{\mathbf{r}}| \sin(90°) \left(\hat{\mathrm{d}\mathbf{l}} \times \hat{\mathbf{r}}\right) = (\mathrm{d}l)(1)(1)(\hat{\mathbf{z}}) = \mathrm{d}l\,\hat{\mathbf{z}}.$$

(5) We factor out all of the constants from the integrals.

(6) Each integral is just the summing of all the infinitemal lengths along the segment in question, to give the total length of the segment. Since each segment is a semicircle of radius R, each integral is a semicircumference: $\frac{1}{2}(2\pi R) = \pi R$.

(7) We simplify the expression and write the magnetostatic constant in terms of μ_0.

(8) Full simplification.

So, our answer agrees with our initial observations (and the answer in the back of the book)!

$$\boxed{\mathbf{B} = \left\{\frac{\mu_0 I}{8R},\ \text{out of the page.}\right\}}$$

Success.

References

[1] Douglas C. Giancoli: *Physics for Scientists & Engineers with Modern Physics, Third Edition*, Prentice Hall (2000)