Physics 1B: Problem 89 from Chapter 13 of Young & Freedman

Pendulum Collision and Oscillation

89. In Fig. 1 the upper ball is released from rest, collides with the stationary lower ball, and sticks to it. The strings are both 50.0 cm long. The upper ball has mass 2.00 kg, and it is initially 10.0 cm higher than the lower ball, which has mass 3.00 kg. Find the frequency and maximum angular displacement of the motion after the collision.

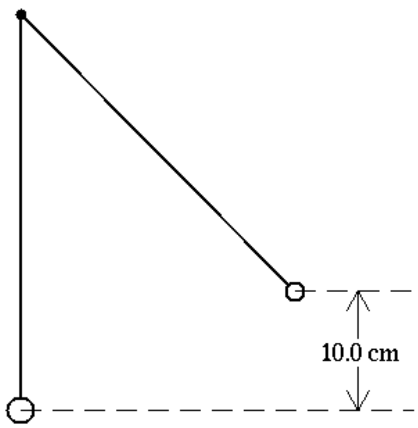

Figure 1: Pendula before collision.

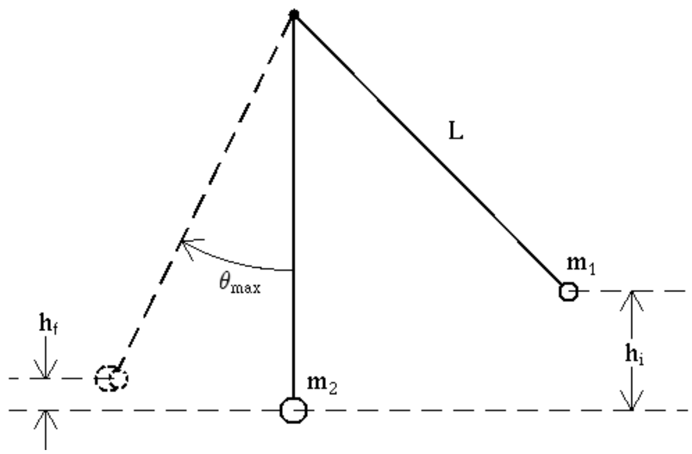

Figure 2: Pendula quantity labels.

Given

$L = (50.0\text{ cm}), \quad m_1 = (2.00\text{ kg}), \quad m_2 = (3.00\text{ kg}), \quad h_\text{i} = (10.0\text{ cm})$

Initial observations

So the upper ball (ball 1) will collide with the lower one (ball 2) and they will swing back and forth (forever, assuming no friction, air resistance, etc.). The question asks for the frequency f of the motion after the collision, and if this motion happens to be simple harmonic, then we will be able to use the formula from the chapter: $f = (1/2\pi)\sqrt{g/L}$. The motion is approximately simple harmonic if the maximum angular displacement θ_max is small, that is, if $\sin\theta_\text{max} \approx \theta_\text{max}$, where θ_max is expressed in radians. So we should solve for θ_max before finding f.

Since the balls stick together, this collision is completely inelastic, which means that mechanical energy is not conserved in the collision. (Some of the mechanical energy is transferred into deforming and heating the balls.) Thus we can only use conservation-of-mechanical-energy before and after the collision. Through the collision, we may use conservation-of-momentum instead. (We can also solve for the mechanical energy lost in the collision to chemical and thermal energy.)

Applying Principles and Algebra

The moment the ball 1 is released, there is no kinetic energy and the initial mechanical energy E_i of the two-pendulum system consists purely of gravitational potential energy; we can set the zero of the gravitational potential energy to be at the height of the lower ball, thus

$$E_{\mathrm{i}} \;=\; m_1 g h_{\mathrm{i}}.$$

By conservation of mechanical energy through the descent of ball 1, the mechanical energy E_{b} just before the collision is equal to E_{i}:

$$\begin{aligned} E_{\mathrm{b}} &= K_{\mathrm{b}} + U_{\mathrm{b}} \;=\; \frac{1}{2} m_1 v^2 + 0 \\ &= E_{\mathrm{i}} \;=\; m_1 g h_{\mathrm{i}}, \end{aligned}$$

where K_{b} and U_{b} are the kinetic and gravitational potential energy and v is the velocity of ball 1 just before the collision. So

$$\begin{aligned} \Rightarrow \qquad \frac{1}{2}\cancel{m_1} v^2 &= \cancel{m_1} g h_{\mathrm{i}} \\ \Rightarrow \qquad v &= \sqrt{2 g h_{\mathrm{i}}}. \end{aligned}$$

Using conservation of momentum through the collision, the system momentum after the collision P_{a} equals the system momentum before the collision P_{b}:

$$\begin{aligned} P_{\mathrm{b}} &= m_1 v \\ &= P_{\mathrm{a}} \;=\; (m_1 + m_2) V, \end{aligned}$$

where V is the velocity of the stuck-pendula after the collision. So

$$\begin{aligned} \Rightarrow \qquad m_1 v &= (m_1 + m_2) V \\ \Rightarrow \qquad V &= \frac{m_1}{m_1 + m_2} v \;=\; \frac{m_1}{m_1 + m_2} \sqrt{2 g h_{\mathrm{i}}}. \end{aligned}$$

Now, using conservation of mechanical energy for the upward swing, and the relationship

$$h_{\mathrm{f}} \;=\; L - L\cos\theta_{\mathrm{max}} \;=\; L(1 - \cos\theta_{\mathrm{max}}),$$

the final mechanical energy is

$$\begin{aligned} E_{\mathrm{f}} &= (m_1 + m_2) g h_{\mathrm{f}} \;=\; (m_1 + m_2) g L (1 - \cos\theta_{\mathrm{max}}) \\ &= E_{\mathrm{a}} \;=\; K_{\mathrm{a}} + U_{\mathrm{a}} \;=\; \frac{1}{2}(m_1 + m_2) V^2 + 0 \;=\; \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 (2 g h_{\mathrm{i}}), \end{aligned}$$

where E_{a}, K_{a} and U_{a} are the mechanical, kinetic, and gravitational potential energies just after the collision. So

$$\begin{aligned} \Rightarrow \qquad \cancel{(m_1 + m_2)}\cancel{g} L (1 - \cos\theta_{\mathrm{max}}) &= \frac{1}{\cancel{2}}\cancel{(m_1 + m_2)}\left(\frac{m_1}{m_1 + m_2}\right)^2 (\cancel{2}\cancel{g} h_{\mathrm{i}}) \\ \Rightarrow \qquad (1 - \cos\theta_{\mathrm{max}}) &= \left(\frac{m_1}{m_1 + m_2}\right)^2 \frac{h_{\mathrm{i}}}{L} \\ \Rightarrow \qquad \cos\theta_{\mathrm{max}} &= 1 - \left(\frac{m_1}{m_1 + m_2}\right)^2 \frac{h_{\mathrm{i}}}{L} \end{aligned}$$

$$\Rightarrow \qquad \theta_{\text{max}} = \cos^{-1}\left[1 - \left(\frac{m_1}{m_1+m_2}\right)^2 \frac{h_{\text{i}}}{L}\right] = \cos^{-1}\left[1 - \left(\frac{2.00 \text{ kg}}{2.00 \text{ kg} + 3.00 \text{ kg}}\right)^2 \frac{(10.0 \text{ cm})}{(50.0 \text{ cm})}\right]$$
$$= 0.254 \text{ rad} \qquad (14.5^\circ)$$

Now that we've found the maximum angular displacement, we can ask, is $\sin\theta_{\text{max}} \approx \theta_{\text{max}}$?

$$\sin\theta_{\text{max}} = \sin(0.254 \text{ rad}) = 0.251 \approx 0.254 = \theta_{\text{max}} \quad \checkmark$$

(Remember that radians and degrees are dimensionless units.) So the motion after the collision is approximately simple harmonic, and we have

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{(9.81 \text{ m/s}^2)}{(0.5000 \text{ m})}} = 0.705 \text{ Hz}.$$

So the frequency and maximum angular displacement of the motion after the collision are

$$\boxed{f = 0.705 \text{ Hz} \qquad \theta_{\text{max}} = 0.254 \text{ rad} \quad (14.5^\circ)}$$

Final observation

Just to show that mechanical energy was actually lost in the collision, we can calculate that lost energy E_{lost}.

$$\begin{aligned}
E_{\text{lost}} &= -\Delta E = -(E_{\text{a}} - E_{\text{b}}) = E_{\text{b}} - E_{\text{a}} \\
&= m_1 g h_{\text{i}} - \frac{1}{2}(m_1+m_2)\left(\frac{m_1}{m_1+m_2}\right)^2 (2gh_{\text{i}}) \\
&= m_1 g h_{\text{i}} - \frac{{m_1}^2}{m_1+m_2} g h_{\text{i}} \\
&= m_1 g h_{\text{i}} \left(1 - \frac{m_1}{m_1+m_2}\right) \\
&= (2.00 \text{ kg})(9.81 \text{ m/s}^2)(0.500 \text{ m})\left(1 - \frac{2.00 \text{ kg}}{2.00 \text{ kg} + 3.00 \text{ kg}}\right) \\
&= 5.89 \text{ J}
\end{aligned}$$

References

[1] Hugh D. Young, Roger A. Freedman: *Sears and Zemansky's University Physics, 12th Edition, Volume 1*, Pearson Addison-Wesley (2008)