Notes for Physics 1B

Gauss's Law

1 Statement of Gauss's Law

The integral form of Gauss's law[1] is

$$\boxed{\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{a} \;=\; \frac{Q_{\mathrm{encl}}}{\varepsilon_0},}$$

where $\mathbf{E}$ is the total electric field, $\mathrm{d}\mathbf{a}$ is an infinitesimal surface area element (making the integral a surface integral), S is a closed surface (as indicated by the circle in the integral sign) over which the integral is performed (so the surface integral is definite with domain S), Q_{encl} is the total charge enclosed[2] by the surface S, and $\varepsilon_0 = (8.85\times 10^{-12}\ \mathrm{C}^2\cdot \mathrm{s}^2/\mathrm{kg}\cdot \mathrm{m}^5)$ is the permittivity of free space. The element $\mathrm{d}\mathbf{a}$ is a vector that is, by convention, directed outward with respect to the surface S. We can write the integral in a slightly more explicit form:

$$\oiint_S \mathbf{E}(\mathbf{r}')\cdot \mathrm{d}\mathbf{a}(\mathbf{r}') \;=\; \frac{Q_{\mathrm{encl}}}{\varepsilon_0}.$$

In this form we see that the integral is really a double integral (since a surface is two-dimensional) and each factor of the integrand $\mathbf{E}\cdot\mathrm{d}\mathbf{a}$ may depend on the location $\mathbf{r}'$ on the surface. We can write Gauss's law in a less explicit form provided that we know the conventional definition of a quantity called flux[3]. We define $\Phi_{V,S}$, the flux of the vector field[4] $\mathbf{V}$ over a surface S, by the scalar surface integral of $\mathbf{V}$ over S:

$$\Phi_{V,S} \;\equiv\; \iint_S \mathbf{V}\cdot \mathrm{d}\mathbf{a}.$$

Note that the surface S *does not* have to be closed, in general. So $\Phi_{E,S}$ is defined in this way using the electric field $\mathbf{E}$ and is called "electric flux". Since we call a surface that is closed and used in Gauss's law a "Gaussian surface", if S is a Gaussian surface we have

$$\boxed{\Phi_{E,S} \;=\; \frac{Q_{\mathrm{encl}}}{\varepsilon_0}.}$$

Note that the subscript "encl" implicitly refers to a closed surface S. We could use a more explicit notation such as Q_S or $Q_{\mathrm{encl},S}$. Similarly, we might occasionally use a less explicit notation such as Φ_E instead of $\Phi_{E,S}$, if the surface S is understood.

[1] There are other Gauss's laws that look just like this, such as the magnetic Gauss's law and the gravitational Gauss's law, but if no adjectives are given then it is assumed to be the electric Gauss's law. Also, there is a differential form of Gauss's law.

[2] I use "encl" for enclosed; later, I'll use "enc" for encircled, in Ampère's law.

[3] For future reference, note that there are two common uses of the word flux in physics: the definition given here is commonly used in electromagnetism, but another definition is used in transport phenomena.

[4] In physics, we use the word "field" to mean a function whose domain is real, physical space (as opposed to some mathematical "space"). So a field has a value at every point in space. A vector field has a vector at every point in space.

2 Uses of Gauss's Law

- Common Problems
  - Find $\Phi_{E,S}$.
    Given (or able to calculate): Q_{encl}.
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    Given (or able to calculate): $\Phi_{E,S}$.
  - Find $\mathbf{E}$.
    Given (or able to calculate):
    * charge or charge density with spherical symmetry.
    * charge or charge density with infinite (or approximately infinite) cylindrical symmetry.
    * charge or charge density with infinite (or approximately infinite) planar symmetry.
    * two or more charges or charge distributions with a special symmetry (so you can use superposition of the two or more individual electric fields to get the total field).
- Less Common Problems
  - Find $\Phi_{E,S}$, the electric flux over a surface S that is *not* closed.
    Given (or able to calculate): Q_{encl}, the net charge enclosed by a closed surface, of which S is a part, where symmetry exists such that the flux over S is a particular fraction of the net flux.
  - Find A_S, the surface area of a closed surface S.
    Given (or able to calculate): Q_{encl}, and E, the magnitude of the elctric field on the surface (which happens to be constant).
  - Find R, the radius of a spherical closed surface S.
    Given (or able to calculate): Q_{encl}, and E, the magnitude of the elctric field on the surface (which happens to be constant).