Physics 1B: Review for Final Exam

Solutions

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In this worksheet we review material from the following chapters of Young and Freedman (plus some additional concepts):

- Chapter 13: Periodic Motion
- Chapter 15: Mechanical Waves
- Chapter 16: Sound and Hearing
- Chapter 25: Current, Resistance, and Electromotive Force
- Chapter 26: Direct-Current Circuits

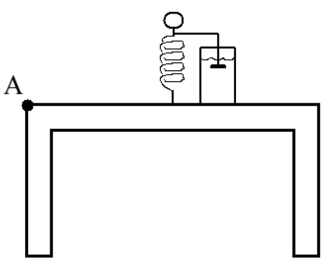
1. *Damped harmonic oscillation of a charged bob.* A spring of relaxed-length L and spring constant k is attached to the center of a tabletop of width W. An arm of negligible weight is attached to the spring with one end immersed in a viscous fluid that provides a damping constant b for oscillation of the spring. A small bob of mass m with charge q (insulated so the charge does not escape) is attached to the spring and held (by you) so that there is no force between the bob and the spring. You let go of the bob at time $t = 0$.

   (a) What is the height y of the bob above the tabletop as a function of time?

   (b) Using what you know about electric fields, what is the magnitude of the electric field at the edge of the tabletop (point A in the 2D drawing) as a function of time?

   (c) What is the magnitude of the electric field at the edge of the tabletop after a very long time?

   Solution

   (a) $y = y(t)$?
   The bob starts off some distance above the equilibrium position y_{eq} of its oscillation. That distance is the initial amplitude of the oscillation A_0. The natural frequency of oscillation is $\omega_{\text{n}} = \sqrt{k/m}$, and the decay rate for the amplitude of oscillation is $r = b/2m$, so the expression for $y(t)$ in these terms is
   $y(t) = y_{\text{eq}} + A_0\, e^{-rt}\, \cos(\omega' t)$, where $\omega' = \sqrt{\omega_{\text{n}}^2 - r^2} = \sqrt{k/m - b^2/4m^2}$.
   (The initial phase ϕ_0 is zero since the oscillation starts at its peak and we're using cosine.)
   Now, what are y_{eq} and A_0 in terms of the given quantities?
   At the equilibrium position the spring is compressed by A_0 so that the spring force pushing upwards equals the gravitational force pulling downwards: $kA_0 = mg$. Thus $A_0 = mg/k$. Since the uncompressed length of the spring is L, the equilibrium height is $y_{\text{eq}} = L - A_0 = L - mg/k$.
   Therefore,
   $$\boxed{y(t) = \big(L - mg/k\big) + \big(mg/k\big)\; e^{-(b/2m)t}\, \cos\Big(\sqrt{k/m - b^2/4m^2}\; t\Big)}$$

   (b) $E = E(t)$ @ A?
   $E = |\mathbf{E}| = K\,|q|\,/r^2 = (1/4\pi\varepsilon_0)\,|q|\,/\,\big[(W/2)^2 + y^2\big]$
   $\boxed{E(t) = |q|\,/\,\Big\{4\pi\varepsilon_0\,\Big[W^2/4 + y^2(t)\Big]\Big\}}$ where $y(t)$ is given in part (a)

   (c) $E(t)$ @ A as $t \to \infty$?

After a long time, as $t \to \infty$, the oscillations die out and $y(t) \to y_{\text{eq}} = L - mg/k$, so

$$\boxed{E(\infty) = |q| \,/\, \left\{4\pi\varepsilon_0 \left[W^2/4 + (L - mg/k)^2\right]\right\}}$$

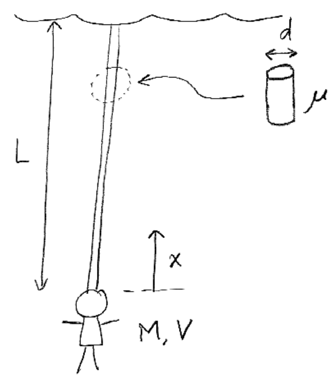
2. *Mechanical wave of variable speed.* (Y&F 15.82) A deep-sea diver is suspended beneath the surface of Loch Ness by a cable of length L that is attached to a boat on the surface. The diver and his suit have a total mass M and a volume V. The cable has a diameter d and a linear mass density μ. The diver thinks he sees something moving in the murky depths and jerks the end of the cable back and forth to send tansverse waves up the cable as a signal to his companions in the boat.

   (a) What is the tension in the cable at its lower end, where it is attached to the diver? Do not forget to include the buoyant force that the water (density ρ_{w}) exerts on him.

   (b) What is the tension in the cable a distance x abover the diver. The buoyant force on the cable must be included in your analysis.

   (c) The speed of transverse waves on the cable is given by $v = \sqrt{F/\mu}$. The speed therefore varies along the cable, since the tension is not constant. (This expression neglects the damping force that the water exerts on the moving cable.) Integrate to find the time required for the first signal to reach the surface.

Solution



(a) F_{T} @ diver?
An unknown force suggests the use of a free-body diagram with Newton's second law of motion. We may select our system to be the diver (and his suit), and the force from the rope on the diver will be the tension F_{T} that we're looking for. Since there is no acceleration, the net force equals zero. Thus $\sum F_x = F_{\text{T}} + F_{\text{buoy}} - F_{\text{g}} = 0$ and $F_{\text{T}} = F_{\text{g}} - F_{\text{buoy}} = Mg - \rho_{\text{w}} V g = \boxed{(M - \rho_{\text{w}} V)g = F_{\text{T}}}$.

(b) F_{T} @ distance x above diver?
We can use the same tactic as in part (a) but declare the system to include the cable up to position x above the diver. Again, since $\sum F_x = 0$, we have $F_{\text{T}} = F_{\text{g}} - F_{\text{buoy}} = M_{\text{tot}} g - \rho_{\text{w}} V_{\text{tot}} g = (M + \mu x)g - \rho_{\text{w}}(V + \pi(d/2)^2 x)g = \boxed{(M - \rho_{\text{w}} V)g + (\mu + \rho_{\text{w}}\pi d^2/4)gx = F_{\text{T}}}$.

(c) Time Δt for signal to reach surface?
To simplify the mathematics let's define a and b such that $F_{\text{T}} = a + bx$.
We have $v = \sqrt{F_{\text{T}}/\mu} = \mu^{-1/2}\sqrt{a + bx}$. We want to know $\Delta t = \int_0^{\Delta t} \mathrm{d}t$. We can relate v to $\mathrm{d}t$ by $v = \mathrm{d}x/\mathrm{d}t$, so

$$\Delta t = \int_0^{\Delta t} \mathrm{d}t = \int_0^L \frac{\mathrm{d}x}{v(x)} = \sqrt{\mu}\int_0^L \frac{\mathrm{d}x}{\sqrt{a + bx}} \qquad \text{and letting } u = a + bx, \text{ so that } \mathrm{d}x = \frac{1}{b}\mathrm{d}u,$$

$$\Delta t = \sqrt{\mu}\frac{1}{b}\int_a^{a+bL} \frac{\mathrm{d}u}{\sqrt{u}} = \frac{\sqrt{\mu}}{b}\left[2\sqrt{u}\right]_a^{a+bL} = \frac{2\sqrt{\mu}}{b}\left[\sqrt{a} - \sqrt{a + bL}\right]$$

$$\boxed{\Delta t = \frac{2\sqrt{\mu}}{(\mu + \rho_{\text{w}}\pi d^2/4)g}\left[\sqrt{(M - \rho_{\text{w}} V)g} - \sqrt{(M - \rho_{\text{w}} V)g + (\mu + \rho_{\text{w}}\pi d^2/4)gL}\right]}$$

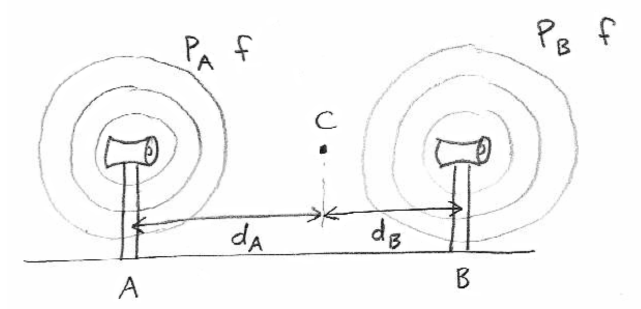
3. *Sound wave interference.* (~Y&F 16.84) Two loudspeakers, A and B, radiate sound uniformly in all directions in air at 20 °C. The acoustic power output from A is 8.00×10^{-4} W, and from B it is 6.00×10^{-5} W. Both loudspeakers are vibrating in phase at a frequency of 172 Hz.

   (a) Determine the difference in phase of the two signals at a point C along the line joining A and B, 3.00 m from B and 4.00 m from A.

   (b) Determine the intensity at point C from speaker A if speaker B is turmed off and the intensity at point C from speaker B if speaker A is turned off.

   (c) With both speakers on, what is the intensity at C?

Solution



(a) $\Delta\phi$ @ C?
Since $v = f\lambda$, $\lambda = v/f = (344 \text{ m/s})/(172 \text{ Hz}) = (2.00 \text{ m}) = \lambda$
Speaker A is 2 wavelengths (4π rad) away from C and speaker B is 1.5 wavelengths (3π rad) away. Thus the phase difference is $\boxed{\Delta\phi = \pi}$, a.k.a. 180°, so the signals are "out of phase".
Using less words and more math, we could calculate $\Delta\phi$ in the following manner:
$\Delta\phi = \phi_\text{A} - \phi_\text{B} = kd_\text{A} - kd_\text{B} = k(d_\text{A} - d_\text{B}) = (2\pi/\lambda)(d_\text{A} - d_\text{B}) = (2\pi f/v)(d_\text{A} - d_\text{B})$
$= \big(2\pi(172 \text{ Hz})/(344 \text{ m/s})\big)\,(4.00 \text{ m} - 3.00 \text{ m}) = \pi$

(b) I_A and I_B @ C?
$I_\text{A} = P_\text{A}/4\pi d_\text{A}^2 = (8.00 \times 10^{-4} \text{ W})/4\pi(4.00 \text{ m})^2 = \boxed{(3.98 \times 10^{-6} \text{ W/m}^2) = I_\text{A}}$ (speaker B off)
$I_\text{B} = P_\text{B}/4\pi d_\text{B}^2 = (6.00 \times 10^{-4} \text{ W})/4\pi(3.00 \text{ m})^2 = \boxed{(5.31 \times 10^{-7} \text{ W/m}^2) = I_\text{B}}$ (speaker A off)

(c) I_AB @ C?
The important conceptual issue is to note that intensities do not add; amplitudes (of displacement or relative pressure) add, and intensity is the square of the amplitude. Also, since the signals are 180° out of phase, while one signal is at its maximum, the other is at its minimum (a negative displacement or relative pressure), so we must subtract the amplitudes:
$I_\text{AB} = (\sqrt{I_\text{A}} - \sqrt{I_\text{B}})^2 = (\sqrt{3.98 \times 10^{-6} \text{ W/m}^2} - \sqrt{5.31 \times 10^{-7} \text{ W/m}^2})^2$
$= \boxed{(1.60 \times 10^{-6} \text{ W/m}^2) = I_\text{AB}}$

4. *Current and resistance between concentric spheres.* (Y&F 25.64) The region between two concentric conducting spheres with radii a and b is filled with a conducting material with resistivity ρ.

   (a) Show that the resistance between the spheres is given by $R = (\rho/4\pi)(1/a - 1/b)$.

   (b) Derive an expression for the current density as a function of radius, in terms of the potential difference V_{ab} between the spheres.

   (c) Show that the result in part (a) reduces to $R = \rho L/A$ when the separation $L = b - a$ between the spheres is small.

Solution

(a) Show $R = (\rho/4\pi)(1/a - 1/b)$.
For simple geometries we use $R = \rho L/A$, where L is the length of the resistor along which the charges travel and A is the cross-sectional area perpendicular to the direction of charge travel, but this case is not so simple. If we assume that charge flows radially (either outward or inward) when

this resistor is in use, then this situation is as if we had put many resistors in a radial pattern, connecting the conductor of radius a to the conductor of radius b, with the resistors all in parallel. Equivalently, we can think of this situation as if we had placed many thin, resistive spherical shells concentrically upon each other, so that they act as resistors in series. Let's use both of these pictures to make two equivalent calculations.

- Series: Although these concentric shells are perhaps conceptually unfamiliar, the calculation is simple. For a thin spherical shell of infinitesimal thickness $\mathrm{d}r$, $\mathrm{d}r$ is the length along which the charges travel and $A = 4\pi r^2$ is the area through which they travel, so the infinitesimal resistance of the shell is $\mathrm{d}R = \rho\,\mathrm{d}r/4\pi r^2$. Thus the total resistance is (since resistances in series add or integrate)

$$R = \int \mathrm{d}R = \int_a^b \rho \frac{\mathrm{d}r}{4\pi r^2} = \frac{\rho}{4\pi}\int_a^b \frac{\mathrm{d}r}{r^2} = \frac{\rho}{4\pi}\left[-\frac{1}{r}\right]_a^b = \frac{\rho}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) \quad \checkmark$$

- Parallel: Although the picture of radially arranged resistors is made of conceptually familiar resistors (unlike the spherical shell resistors), the calculation is quite unusual. (I've actually never seen anything written like this before...I'm just showing this to you for completeness. You should never have to do anything like this...this is crazy.)

$$\mathrm{d}\left(\frac{1}{\mathrm{d}\left(\frac{1}{R}\right)}\right) = \rho\frac{\mathrm{d}r}{\mathrm{d}a} = \frac{\rho\,\mathrm{d}r}{r\mathrm{d}\theta\, r\sin\theta\mathrm{d}\phi}$$

$$\frac{1}{\mathrm{d}\left(\frac{1}{R}\right)} = \int \mathrm{d}\left(\frac{1}{\mathrm{d}\left(\frac{1}{R}\right)}\right) = \int_a^b \frac{\rho\,\mathrm{d}r}{r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi} = \frac{\rho}{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}\int_a^b\frac{\mathrm{d}r}{r^2} = \frac{\rho}{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\Rightarrow \qquad \mathrm{d}\left(\frac{1}{R}\right) = \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\frac{1}{R} = \int \mathrm{d}\left(\frac{1}{R}\right) = \iint_{\text{all angles}} \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi$$

$$= \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}[-\cos\theta]_0^\pi\,[\phi]_0^{2\pi} = \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}(1+1)(2\pi-0) = \rho^{-1}\left(\frac{1}{a}-\frac{1}{b}\right)^{-1}4\pi$$

$$\Rightarrow \qquad R = \frac{\rho}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) \quad \checkmark$$

(b) $J = J(r, V_{ab})$?
$V_{ab} = IR$ and $J = I/A(r) = I/4\pi r^2$

$$\Rightarrow \qquad J(r) = \frac{(V_{ab}/R)}{4\pi r^2} = \frac{V_{ab}}{4\pi r^2}\frac{4\pi}{\rho\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{V_{ab}}{\rho\left(\frac{1}{a}-\frac{1}{b}\right)r^2}\frac{ab}{ab} = \boxed{\frac{V_{ab}\,ab}{\rho(b-a)r^2} = J(r)}$$

(c) Show $R \to \rho L/A$ for $L = b - a$ small enough.

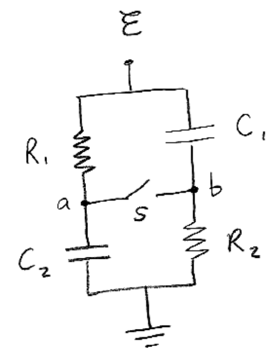
$$R = \frac{\rho}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{\rho(b-a)}{4\pi ab} = \frac{\rho L}{4\pi ab} = \frac{\rho L}{4\pi a(a+L)} \approx \frac{\rho L}{4\pi a^2} = \frac{\rho L}{A} \quad \checkmark$$

where the approximation is true if $L \ll a$ such that $a + L \approx a$.

5. *Voltages in an RC circuit with a switch.* (Y&F 26.74)

   (a) What is the potential of point a with respect to point b in the figure below when switch S is open?
   (b) Which point, a or b is at higher potential?
   (c) What is the final potential of point b with respect to ground when switch S is closed?
   (d) How much does the charge on each capacitor change when S is closed?

   Solution

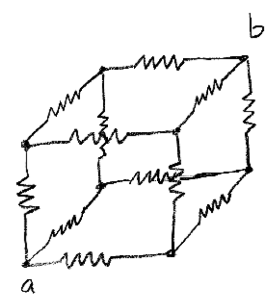
   (a) V_{ab} with switch open?
   Each capacitor becomes fully charged and there is no longer current in the resistors.
   $\Rightarrow V_{\text{ab}} = V_{\text{a}} - V_{\text{b}} = \mathcal{E} - 0 = \boxed{\mathcal{E} = V_{\text{ab}}}$
   (b) Point a or b @ higher potential?
   Point a is connected to the source and point b is connected to ground ($V = 0$). Given $\mathcal{E} > 0$, $\boxed{\text{point a}}$ is at a higher potential.
   (c) V_{b} long after switch closed?
   Charge will always run through the resistors and the capacitors will become charged according to the potentials across the resistors. The current in the resistors is $I = \mathcal{E}/(R_1 + R_2)$, so $V_{\text{b}} = IR_2 = \boxed{\mathcal{E}R_2/(R_1 + R_2) = V_{\text{b}}}$.
   (d) ΔQ of each capacitor after switch closed?

$$C_1: \quad \Delta Q_1 = C_1(IR_1) - C_1\mathcal{E} = C_1\frac{\mathcal{E}}{R_1 + R_2}R_1 - C_1\mathcal{E} = \boxed{-C_1\mathcal{E}\left(1 - \frac{R_1}{R_1 + R_2}\right) = \Delta Q_1}$$

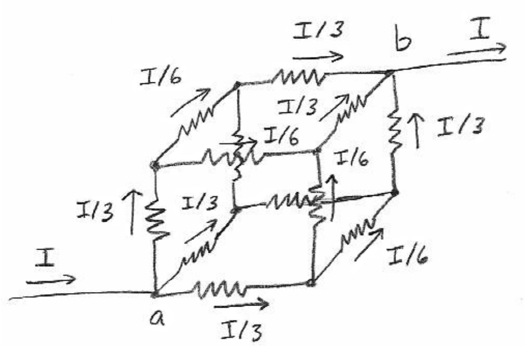
$$C_2: \quad \Delta Q_2 = C_2(IR_2) - C_2\mathcal{E} = C_2\frac{\mathcal{E}}{R_1 + R_2}R_2 - C_2\mathcal{E} = \boxed{-C_2\mathcal{E}\left(1 - \frac{R_2}{R_1 + R_2}\right) = \Delta Q_2}$$

6. *Equivalent resistance of a cube frame of resistors.* (Y&F 26.92) Suppose a resistor R lies along each edge of a cube (12 resistors in all) with connections at the corners. Find the equivalent resistance between two diagonally opposite corners of the cube (points a and b in the figure).

   Solution

   We can use the symmetry of the cube and imagine charge flowing to get a ratio between voltage and current. Charge flowing through a should split evenly among the three branches it meets since there's nothing to distinguish them. Those three branches, and the three branches meeting point b, should each have current $I/3$. The same argument shows that the six remaining branches have current $I/6$.

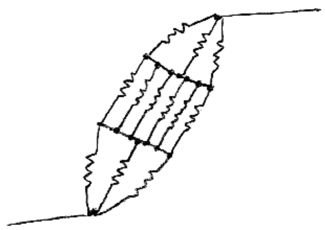
   If we pick any path from b to a, we get

$$V_{ab} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}IR$$

   Thus we have

$$R_{\text{eff}} = \frac{V_{ab}}{I} = \boxed{\frac{5}{6}R = R_{\text{eff}}}$$

We can also see that, effectively, the ends of the three branches attached to a are at the same potential since the resistances, and voltages across them, are the same. The last drawing shows that this implies that the arrangement is effectively like three resistors in parallel, in series with six resistors in parallel, in series with three resistors in parallel.



$$
\begin{aligned}
R_{\text{eff}} &= \frac{1}{\frac{1}{R}+\frac{1}{R}+\frac{1}{R}} + \frac{1}{\frac{1}{R}+\frac{1}{R}+\frac{1}{R}+\frac{1}{R}+\frac{1}{R}+\frac{1}{R}} + \frac{1}{\frac{1}{R}+\frac{1}{R}+\frac{1}{R}} \\
&= \frac{1}{\frac{3}{R}} + \frac{1}{\frac{6}{R}} + \frac{1}{\frac{3}{R}} \\
&= \frac{1}{3}R + \frac{1}{6}R + \frac{1}{3}R \\
&= \frac{5}{6}R
\end{aligned}
$$