Physics 1B: E-Field from a Point-Source Integral

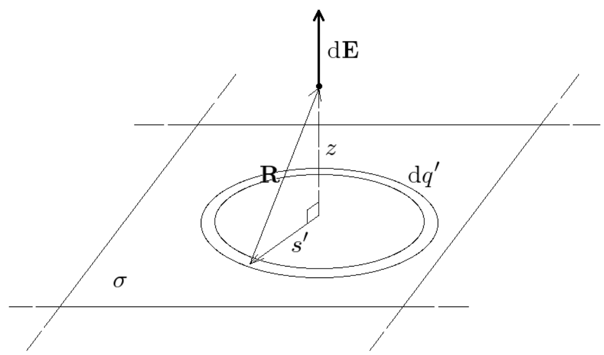
E-Field of a Plane of Charge

1 Problem

Using the remote-point-source expression for the electric field (rather than Gauss's law), solve for the electric field everywhere surrounding an infinite plane with surface charge density σ.

2 Quick Solution

By symmetry, $\mathbf{E}$ only points toward or away from the plane, and it can only depend on z.



$$\begin{aligned}
\mathbf{E}(z) &= K_\mathrm{e} \int\limits_{\text{distr.}} \frac{\mathrm{d}q'}{R^2}\hat{\mathbf{R}} \;=\; K_\mathrm{e} \int\limits_{\text{distr.}} \frac{\mathrm{d}q'}{R^3}\mathbf{R} \\
&= K_\mathrm{e} \int\limits_{\text{plane}} \frac{\mathrm{d}q'}{R^3} R_z \hat{\mathbf{z}} \\
&= K_\mathrm{e} \int_0^\infty \frac{\sigma 2\pi s' \mathrm{d}s'}{\left[s'^2 + z^2\right]^{3/2}} z\hat{\mathbf{z}} \\
&= 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \int_0^\infty \frac{s' \mathrm{d}s'}{\left[s'^2 + z^2\right]^{3/2}} \\
&= 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \int_{z^2}^\infty \frac{\frac{1}{2}\mathrm{d}u}{u^{3/2}} \;=\; 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \left[-\frac{1}{u^{1/2}}\right]_{z^2}^\infty \;=\; 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \left[0 - \left(-\frac{1}{|z|}\right)\right] \\
&= \pm 2\pi K_\mathrm{e} \sigma \hat{\mathbf{z}} \quad (+ \text{ sign for } z > 0, \; - \text{ sign for } z < 0) \\
&= \pm 2\pi \left(\frac{1}{4\pi\varepsilon_0}\right) \sigma \hat{\mathbf{z}} \;=\; \pm \frac{\sigma}{2\varepsilon_0} \hat{\mathbf{z}}
\end{aligned}$$

We see that the electric field in fact does not depend on z after all.

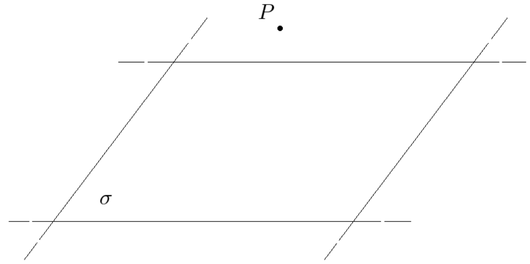
$\mathbf{E} = \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points away from the plane,

$\mathbf{E} = \mathbf{0}$ inside the plane.

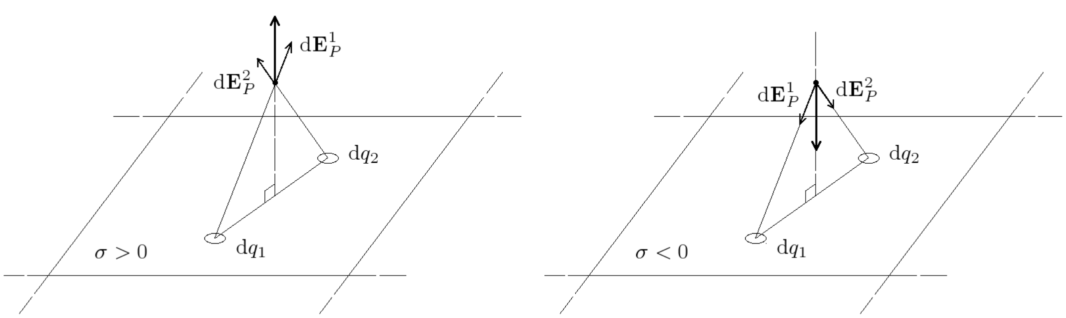
3 Explained Solutions

3.1 Qualitative Analysis

As with every problem, we should make a qualitative analysis before getting quantitative; we can get a sense of what the answer should be and potentially simplify the calculation with our conceptual ideas. A good way to start is to draw a picture:



We can arbitrarily choose to draw the plane in a horizontal orientation. Let's pick a random point P above the plane and see if we can reason what direction the electric field ("E-field") $\mathbf{E}_P$ at this point is pointing. Let's assume for now that the charge density is positive, $\sigma > 0$. If we examine a small patch on the plane, it will have a small positive amount of charge $\mathrm{d}q_1$ and that will contribute a small amount of E-field $\mathrm{d}\mathbf{E}_P^1$ at the point P that points directly away from the patch. (See the illustration on the left, below.) If we look at another patch that is symmetrically placed on the other side of P, it will contribute its own small amount of E-field $\mathrm{d}\mathbf{E}_P^2$, and the magnitudes of these two contributions are the same, $\mathrm{d}E_P^1 = \mathrm{d}E_P^2$, since the source charges $\mathrm{d}q_1$ and $\mathrm{d}q_2$ are the same distance from point P. The geometry of this situation is such that the horizontal components of these two E-field vectors cancel, but the vertical components add to point away from the plane.



Since an infinite plane can be completely constructed from these pairs of patches, it follows that the total electric field at point P should have no horizontal component and only a vertical component pointing away from the plane. This same logic allows us to conclude that for negative charge density, $\sigma < 0$, the electric field must point directly toward the plane, with no component parallel to the plane. (See the illustration on the right, above.)

Another kind of argument gives us this same result. This other argument is a proof by contradiction. Here it is: Since the source charge distribution is symmetrical with respect to translation or shifting along the infinite plane and with respect to rotation within the plane, it must be true that the electric field due to that distribution is also independent of translation along the plane and rotation about an axis perpendicular to the plane. Let's put it this way – if you examine this plane, look away from it, and someone rotates and translates it in an arbitrary way, when you look back you won't be able to tell it's been moved. So the electric field generated by this moved plane of charge must look the same too. But if the E-field at P has a horizontal component, it is not symmetrical with respect to rotations; rotating this vector gives a noticeably different vector. Thus the E-field cannot have a horizontal component. And, of course, since the E-field points away from positive charge and toward negative charge, the field points away from the plane if $\sigma > 0$ and toward the plane if $\sigma < 0$.

These qualitative arguments allow us to deduce the direction of the electric field but not the magnitude. And we haven't deduced how the strength of the field depends on the distance from the plane. Let's do that now, quantitatively.

3.2 Quantitative Analysis

The "remote-point-source expression" for the electric field can be written in the following ways:

Point Charge	Differential Form	Integral Form
$\mathbf{E} = K_\mathrm{e}\dfrac{q}{R^2}\hat{\mathbf{R}}$	$\mathrm{d}\mathbf{E} = K_\mathrm{e}\dfrac{\mathrm{d}q}{R^2}\hat{\mathbf{R}}$	$\mathbf{E} = K_\mathrm{e}\displaystyle\int_{\text{distr.}} \dfrac{\mathrm{d}q}{R^2}\hat{\mathbf{R}}$

The vector $\mathbf{R}$ with magnitude R and direction $\hat{\mathbf{R}}$ always goes from the source charge to the point of interest (the location of the E-field that you are calculating). The integral sign with "distr." means you integrate over a distribution of charge to include the effect of all infinitesimal pieces $\mathrm{d}q$ in the distribution. The integral form is simply $\mathbf{E} = \int \mathrm{d}\mathbf{E}$.

Instead of using $\hat{\mathbf{R}}/R^2$ in these calculations, it is oftentimes easier to use $\mathbf{R}/R^3$; these are equivalent since $\mathbf{R} = R\hat{\mathbf{R}}$:

$$\frac{\hat{\mathbf{R}}}{R^2} = \frac{\left(\frac{\mathbf{R}}{R}\right)}{R^2} = \frac{\mathbf{R}}{R^3}.$$

This simply reduces your calculations by one step, doing symbolically what you might have tried to do with larger messier expressions.

3.2.1 Using Symmetry

We can use the symmetry that we observed qualitatively above to simplify the calculations; we can simply ignore the horizontal components of the electric field, since we know they must be zero. We can use the rotational symmetry to select cylindrical coordinates, and this also reduces the amount of calculation. (See the next section, 3.2, for a calculation done fully in rectilinear coordinates, where all symmetries are ignored.) We expect the E-field to depend on vertical position z, but it's independent of horizontal position (x, y). So let's start with a diagram (or two), to define all our quantities, and then we can perform the integral. Explanations of each step are listed below the calculation, after the boxed solution.

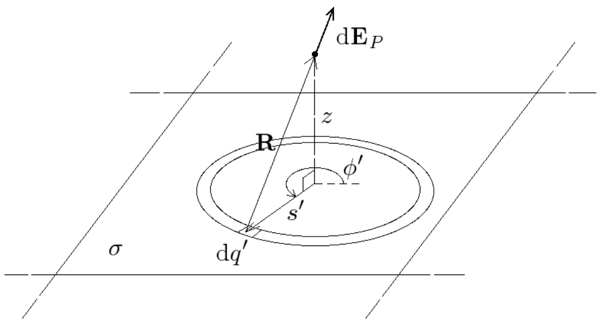


Figure 1: We'll use this diagram to perform the integral. Here, $\mathrm{d}q' = \sigma s' \mathrm{d}\phi' \mathrm{d}s'$.

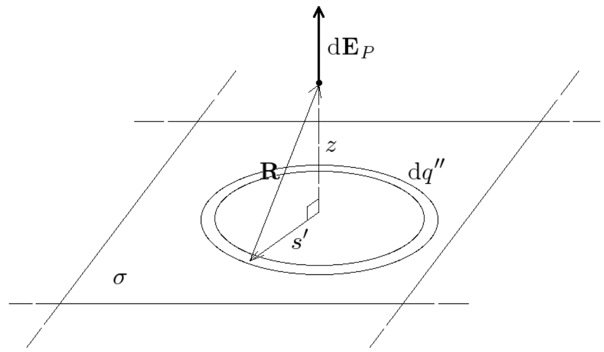


Figure 2: We could use this diagram to be a bit quicker. Here, $\mathrm{d}q'' = \sigma 2\pi s' \mathrm{d}s'$.

$$
\begin{aligned}
\mathbf{E}_P(z) &= K_\mathrm{e} \int\limits_{\text{distr.}} \frac{\mathrm{d}q'}{R^2}\hat{\mathbf{R}} && (1)\\
&= K_\mathrm{e} \int\limits_{\text{distr.}} \frac{\mathrm{d}q'}{R^3}\mathbf{R} && (2)\\
&= K_\mathrm{e} \iint\limits_{\text{plane}} \frac{\mathrm{d}q'}{R^3} R_z \hat{\mathbf{z}} && (3)\\
&= K_\mathrm{e} \int_0^\infty \int_0^{2\pi} \frac{\sigma s' \mathrm{d}\phi' \mathrm{d}s'}{\left[s'^2 + z^2\right]^{3/2}} z\hat{\mathbf{z}} && (4)\\
&= K_\mathrm{e} \sigma z \hat{\mathbf{z}} \left(\int_0^{2\pi} \mathrm{d}\phi' \right) \left(\int_0^\infty \frac{s' \mathrm{d}s'}{\left[s'^2 + z^2\right]^{3/2}} \right) && (5)\\
&= K_\mathrm{e} \sigma z \hat{\mathbf{z}} (2\pi) \left(\int_{z^2}^\infty \frac{\frac{1}{2}\mathrm{d}u}{u^{3/2}} \right) && (6)\\
&= 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \left[-\frac{1}{u^{1/2}} \right]_{z^2}^\infty && (7)\\
&= 2\pi K_\mathrm{e} \sigma z \hat{\mathbf{z}} \left[0 - \left(-\frac{1}{|z|} \right) \right] && (8)\\
&= \pm 2\pi K_\mathrm{e} \sigma \hat{\mathbf{z}} \qquad (+ \text{ sign for } z > 0,\ - \text{ sign for } z < 0) && (9)\\
&= \pm 2\pi \left(\frac{1}{4\pi\varepsilon_0} \right) \sigma \hat{\mathbf{z}} && (10)\\
&= \pm \frac{\sigma}{2\varepsilon_0} \hat{\mathbf{z}} && (11)
\end{aligned}
$$

Very interesting: we've found that the strength of the electric field doesn't depend on distance from the plane. The $\hat{\mathbf{R}}/R^2$ geometry and the planar geometry somehow offset each other exactly, so the electric field doesn't depend on what point P we pick at all, other than whether it's above or below the plane. So the solution can be written as

$$\boxed{\mathbf{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}} & \text{above the plane} \\ -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}} & \text{below the plane} \end{cases}}$$

or

$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \quad \text{where } \hat{\mathbf{n}} \text{ points away from the plane.}}$$

It would make sense that $\mathbf{E} = \mathbf{0}$ inside the plane, where the charge is, but this is not clear from the mathematics.

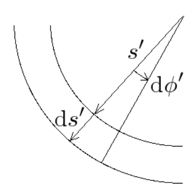
Explanation of each step

(1) Since we have a distribution of charge, we use the integral form of the remote-point-source expression for the electric field at point P. The variables dq' and $\mathbf{R}$ (an thus R and $\hat{\mathbf{R}}$) are defined in Figure 1. The z-dependence of $\mathrm{d}\mathbf{E}_P$ can arise because R and $\hat{\mathbf{R}}$ depend on the vertical position z of the point P. We are neglecting x- and y-dependence in the drawing and equation since we reasoned by translational symmetry there should be no such dependence.

(2) As I noted above, this step of expressing $\hat{\mathbf{R}}$ in terms of $\mathbf{R}$ and R simplifies the calculation.

(3) Here we use the observed rotational symmetry to drop the horizontal x- and y- components of the integral. We also show for clarity that the integral over the planar distribution of charge will be a double integral, since a plane is two-dimensional.

(4) We must pick explicit variables of integration to perform this integral, so we choose cylindrical coordinates (s', ϕ') due to the rotational symmetry. We have to express everything in terms of s' and ϕ' (and $\mathrm{d}s'$ and $\mathrm{d}\phi'$) and write the limits of integration. Since we integrate over the whole plane, we integrate over ϕ' from 0 to 2π and over s' from 0 to ∞. Using the diagram in Figure 1 or the sketch below, we can see that $\mathrm{d}q'$ is the charge in the infinitesimal cylindrical-coordinate-box with sides of length $s'd\phi'$ on the inner edge, $\mathrm{d}s'$ on the radial edges, and $(s'+\mathrm{d}s')\mathrm{d}\phi'$ on the outer edge. Since we can neglect squares of infinitesimals, the area of the box is $\mathrm{d}a' = s'\mathrm{d}\phi'\mathrm{d}s'$, no matter how you calculate its area. Since σ is the charge per area, the charge is $\mathrm{d}q' = \sigma \mathrm{d}a' = \sigma s'\mathrm{d}\phi'\mathrm{d}s'$.



The z-component of $\mathbf{R}$ is $R_z = z$ and, using the Pythagorean theorem, the length of $\mathbf{R}$ is $R = \sqrt{s'^2 + z^2}$.

(5) We pull out the constants from the integral, including the constant unit vector $\hat{\mathbf{z}}$, and separate the integrals, since they are independent. (By the way, we could have skipped doing the ϕ' integral explicitly by adopting the diagram in Figure 2 and using $\mathrm{d}q'' = 2\pi s'\mathrm{d}s'$, as we did in the Quick Solution above.)

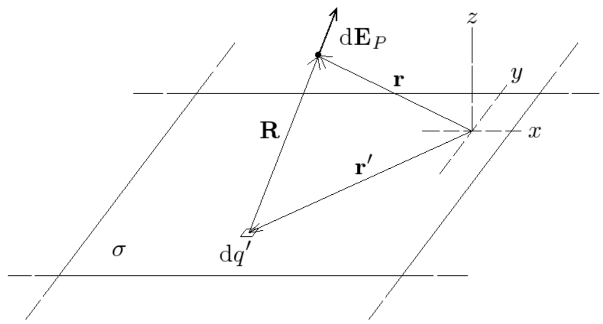
(6) For the s'-integral, we use the substitution $u = s'^2 + z^2$. So $\mathrm{d}u = 2s'\mathrm{d}s'$, and $s'\mathrm{d}s' = \frac{1}{2}\mathrm{d}u$. We change the limits of integration since when $s' = 0$, $u = z^2$, and when $s' = \infty$, $u = \infty^2 + z^2 = \infty$.

(7) To perform this integral, we remember the derivative $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$, thus the integral is $\int x^m = \frac{1}{m+1}x^{m+1}$, so long as $m \neq -1$.

(8-11) Simplifications, using $K_\mathrm{e} = 1/(4\pi\varepsilon_0)$ in step 10 to decrease the number of symbols.

3.2.2 Disregarding Symmetry

If we disregard the symmetry we observed and perform the integral through brute force, we work harder for the same answer. At least this procedure will allow us to practice our integral tricks, but physics at its best allows you to know a lot while being as lazy as possible – that's efficiency!



- The point P is at a random position (x, y, z), which can be expressed several ways as a vector: $\mathbf{r} = \langle x, y, z\rangle = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. The source charge $\mathrm{d}q'$ is at the position (x', y', z'), where $z' = 0$, or $\mathbf{r}' = \big\langle x', y', 0\big\rangle = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$.
- $\mathbf{R} = \mathbf{r} - \mathbf{r}' = \big\langle x - x', y - y', z\big\rangle$ and, using the three dimensional Pythagorean theorem $R = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$.
- Since we have decided (for no good reason) to use rectilinear coordinates (and integrate with respect to them), the source charge can be expressed as $\mathrm{d}q' = \sigma \mathrm{d}a' = \sigma \mathrm{d}x'\mathrm{d}y'$.

$$
\begin{aligned}
\mathbf{E}_P(x, y, z) &= K_\mathrm{e} \int\limits_{\text{distr.}} \frac{\mathrm{d}q'}{R^2}\hat{\mathbf{R}} \\
&= K_\mathrm{e} \iint\limits_{\text{plane}} \frac{\mathrm{d}q'}{R^3}\mathbf{R} \\
&= K_\mathrm{e} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sigma \mathrm{d}x'\mathrm{d}y'}{\left[(x - x')^2 + (y - y')^2 + z^2\right]^{3/2}} \Big\langle x - x', y - y', z\Big\rangle \\
&= K_\mathrm{e}\sigma \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} \frac{\mathrm{d}u\,\mathrm{d}v}{\left[u^2 + v^2 + z^2\right]^{3/2}} \langle u, v, z\rangle \\
&= K_\mathrm{e}\sigma \int\limits_{\infty}^{-\infty}\int\limits_{\infty}^{-\infty} \frac{u\,\mathrm{d}u\,\mathrm{d}v}{\left[u^2 + v^2 + z^2\right]^{3/2}}\,\hat{\mathbf{x}} + K_\mathrm{e}\sigma \int\limits_{\infty}^{-\infty}\int\limits_{\infty}^{-\infty} \frac{v\,\mathrm{d}u\,\mathrm{d}v}{\left[u^2 + v^2 + z^2\right]^{3/2}}\,\hat{\mathbf{y}} + K_\mathrm{e}\sigma z \int\limits_{\infty}^{-\infty}\int\limits_{\infty}^{-\infty} \frac{\mathrm{d}u\,\mathrm{d}v}{\left[u^2 + v^2 + z^2\right]^{3/2}}\,\hat{\mathbf{z}}
\end{aligned}
$$

We'll now solve this integral component-by-component. Note that we used the substitutions $u = x - x'$ and $v = y - y'$, where $\mathrm{d}u = -\mathrm{d}x'$ and $\mathrm{d}v = -\mathrm{d}y'$ (since x and y are constant in the integral), so the limits of integration reversed. In the last step above, we simply wrote out the vector $\langle u, v, z\rangle$ in terms of the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ and expanded the integral into the three component terms. Note also, that this substitution is equivalent to changing the origin to the point directly below point P. We could have started out with that origin if we had realized ahead of time it would simplify the calculation.

Looking at the x-component and taking the u-integral first, we have

$$
\mathbf{E}_{Px}(x, y, z) = K_\mathrm{e}\sigma \int\limits_{\infty}^{-\infty} \left(\int\limits_{\infty}^{-\infty} \frac{u\,\mathrm{d}u}{\left[u^2 + v^2 + z^2\right]^{3/2}} \right) \mathrm{d}v
$$

$$= K_e\sigma \int_{\infty}^{-\infty} \left(\int_{\infty}^{\infty} \frac{\frac{1}{2}\mathrm{d}u'}{u'^{3/2}} \right) \mathrm{d}v$$

$$= K_e\sigma \int_{\infty}^{-\infty} (0)\, \mathrm{d}v$$

$$= 0$$

where we used the substitution $u' = u^2+v^2+z^2$ and found that the limits of integration became the same, so the u-integral was zero. Alternatively, we could have noticed on the first line above that the u-integral contained an odd function (u) divided by an even function ($[u^2+v^2+z^2]^{3/2}$) which is, overall, an odd function; and integrating an odd function over a symmetric domain $(-\infty, \infty)$ yields zero. So, by brute force, we have shown that the electric field has no horizontal x-component. We come to the same conclusion for the y-component if we examine the v-integral first. Thus, the horizontal components drop out.

Now for the z-component. See the explanations for each step below.

$$\mathbf{E}_{Pz}(x,y,z) = K_e\sigma z \int_{\infty}^{-\infty} \left(\int_{\infty}^{-\infty} \frac{\mathrm{d}u}{\left[u^2+(v^2+z^2)\right]^{3/2}} \right) \mathrm{d}v \tag{12}$$

$$= K_e\sigma z \int_{\infty}^{-\infty} \left(\int_{\infty}^{-\infty} \frac{\mathrm{d}u}{[u^2+b^2]^{3/2}} \right) \mathrm{d}v \tag{13}$$

$$= K_e\sigma z \int_{\infty}^{-\infty} \left(\int_{\infty}^{-\infty} \left(\frac{1}{h}\right)^3 \mathrm{d}u \right) \mathrm{d}v \tag{14}$$

$$= K_e\sigma z \int_{\infty}^{-\infty} \left(\int_{\pi/2}^{-\pi/2} \left(\frac{\cos\theta}{b}\right)^3 \left(\frac{b\,\mathrm{d}\theta}{\cos^2\theta}\right) \right) \mathrm{d}v = K_e\sigma z \int_{\infty}^{-\infty} \left(\frac{1}{b^2} \int_{\pi/2}^{-\pi/2} \cos\theta\,\mathrm{d}\theta \right) \mathrm{d}v \tag{15}$$

$$= K_e\sigma z \int_{\infty}^{-\infty} \left(\frac{1}{b^2} [\sin\theta]_{\pi/2}^{-\pi/2} \right) \mathrm{d}v = K_e\sigma z \int_{\infty}^{-\infty} \left(\frac{1}{b^2} \big[(-1)-(1)\big] \right) \mathrm{d}v \tag{16}$$

$$= K_e\sigma z \int_{\infty}^{-\infty} \left(-\frac{2}{b^2} \right) \mathrm{d}v = -2K_e\sigma z \int_{\infty}^{-\infty} \frac{\mathrm{d}v}{v^2+z^2} \tag{17}$$

$$= -2K_e\sigma z \int_{\infty}^{-\infty} \left(\frac{1}{h}\right)^2 \mathrm{d}v \tag{18}$$

$$= -2K_e\sigma z \int_{\pi/2}^{-\pi/2} \left(\frac{\cos\theta}{|z|}\right)^2 \left(\frac{|z|\,\mathrm{d}\theta}{\cos^2\theta}\right) = -2K_e\sigma z \frac{1}{|z|} \int_{\pi/2}^{-\pi/2} \mathrm{d}\theta \tag{19}$$

$$= -(\pm)2K_e\sigma\,[\theta]_{\pi/2}^{-\pi/2} = -(\pm)2K_e\sigma\,\big[(-\pi/2)-(\pi/2)\big] = -(\pm)2K_e\sigma(-\pi) \tag{20}$$

$$= \pm 2\pi K_e\sigma = \pm 2\pi \left(\frac{1}{4\pi\varepsilon_0}\right)\sigma = \pm\frac{\sigma}{2\varepsilon_0} \tag{21}$$

The positive sign is for positive z (above the plane), and the negative sign is for negative z (below the plane). So we see that the electric field doesn't depend on x, y, or z:

$$\mathbf{E}_P(x, y, z) = \left\langle 0, 0, \pm\frac{\sigma}{2\varepsilon_0} \right\rangle = \pm\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}$$

So again we find the solution

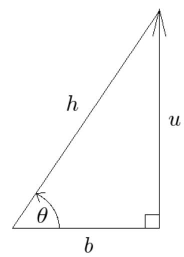
$$\boxed{\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \quad \text{where } \hat{\mathbf{n}} \text{ points away from the plane.}}$$

and again we note that we should have $\mathbf{E} = \mathbf{0}$ on the plane.

Explanation of each step

(12) Taking the u-integral first, we note that since v and z are constant with respect to this integral, we can consider $v^2 + z^2$ to be a constant and rename it to simplify the expression.

(13) We rename the constant $v^2 + z^2$ as b^2, since sums of squares can be geometrically interpreted as a square of another quantity (which happens to be the hypotenuse of a triangle with sides v and z). Since we don't immediately know how to solve this integral we can be inspired by seeing the sum of two squares ($u^2 + b^2$) to start thinking about triangles and whether this integral could be solved in terms of an angle of such a triangle.



(14) Drawing the triangle above, we see the hypotenuse is $h = \sqrt{u^2 + b^2}$. Now we need to rewrite h and $\mathrm{d}u$ in terms of θ and the constant b. The hypotenuse h relates to θ and b by a cosine, $\cos\theta = b/h$, and u relates by a tangent, $\tan\theta = u/b$. So $(1/h) = (\cos\theta/b)$ and $\mathrm{d}u = b\sec^2\theta\,\mathrm{d}\theta = (b\,\mathrm{d}\theta/\cos^2\theta)$.

(15-17) Using the relations derived above, we can perform the resulting simple integral. After simplifying and putting b back as we found it to $v^2 + z^2$, we can be inspired again to think of triangles.

(18-19) If fact, we can think of the same triangle above, except with b rewritten as the constant $|z|$ and u rewritten as v, where $h = \sqrt{v^2 + z^2}$. We make the same substitutions as before, but the resulting integral is even simpler this time.

(20) We will rewrite $z/|z|$ as $\pm$, keeping in mind that the positive sign is for positive z (above the plane), and the negative sign is for negative z (below the plane). We continue to evaluate the integral.

(21) We use $K_\mathrm{e} = 1/(4\pi\varepsilon_0)$ in our simplifications to decrease the number of symbols.