Damped Harmonic Motion

Closing Doors and Bumpy Rides

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Prerequisites and Goal

Assuming you are familiar with simple harmonic motion, its equation of motion, and its solutions, we will now proceed to damped harmonic motion. In this analysis we'll find that among all the possible levels of damping, from zero to infinity, that there is a special "sweet spot" level of damping called critical damping, that will allow you to automatically shut a door most quickly (and smoothly) without slamming it, for example. This principle can apply to shocks in vehicle suspension as well, to quickly absorb impacts from bumps in the road. We'll examine these topics again near the end of this paper.

Qualitative Analysis of Motion

Before getting mathematical, we should examine some real systems and get an intuitive feel for them. Well, on paper, we'll just examine some graphs representing damped harmonic motion with different levels damping and discuss why they make sense. In this section we'll pretend that the graphs are created using data from a real oscillator, but we should realize that real data would actually be a bit more complicated. You can ignore the graphs' Keys for now and refer to them once you've examined the mathematics later in this paper.

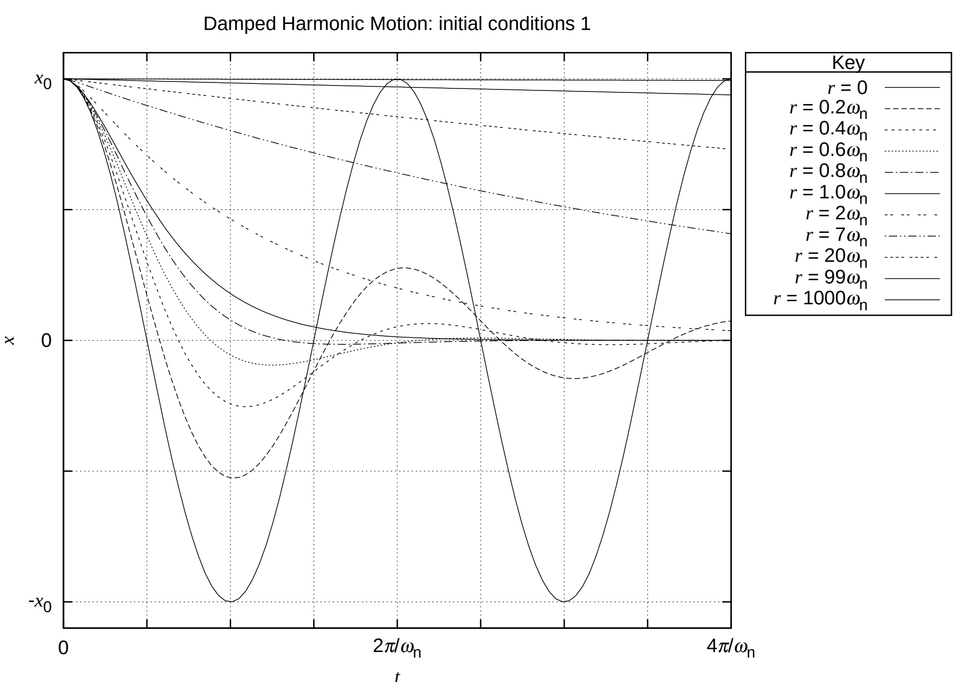

Figure 1: Damped harmonic motion with different levels of damping, with initial conditions 1.

We'll just examine one-dimensional motion in this paper. For concreteness, we'll imagine the oscillator is an object with mass sitting on a frictionless table, attached to a spring (which is attached to a wall) and a damper (which contains some viscous fluid). The variable x will denote its displacement from its equilibrium position. We'll watch the motion of the object under several conditions, where the spring and mass are kept the same but the damping fluid in the damper is made progressively more viscous, starting from zero viscosity. We'll examine two sets of initial conditions; 1) one where the object is released from rest at x_0 (see Figure 1), and 2) one where the object starts at the equilibrium position with a velocity v_0, after being smacked with the appropriate impulse (see Figure 2).

The first noticeable fact from Figure 1 is that there are two distinct kinds of motion: oscillations with some rate of decay and "decay" or relaxation with no oscillations. Explaining the extreme cases in Figure 1 should be most obvious. With no damping, we simply have simple harmonic motion, a sinusoidal motion which is given by an unshifted cosine because of the initial conditions. With infinite damping, the object does not move at all, being "frozen" by an infinitely viscous fluid (that is, a solid) at its initial position, and so its position-versus-time curve is a straight horizontal line. What about the intermediate cases? We might first note that we know intuitively that a damping force tends to take the energy out of a system, eventually leaving it without motion and with an unstressed spring. As the damping is increased, we see that the object moves more slowly towards the equilibrium position, and when oscillating, its period of oscillation is increased since it's moving more slowly. It seems that as the damping increases, the period increases more and more rapidly, apparently going to infinity at some point, at which point the curve becomes pure relaxation. Then, as the damping continues to increase, the relaxation just takes longer and longer to reach equilibrium.

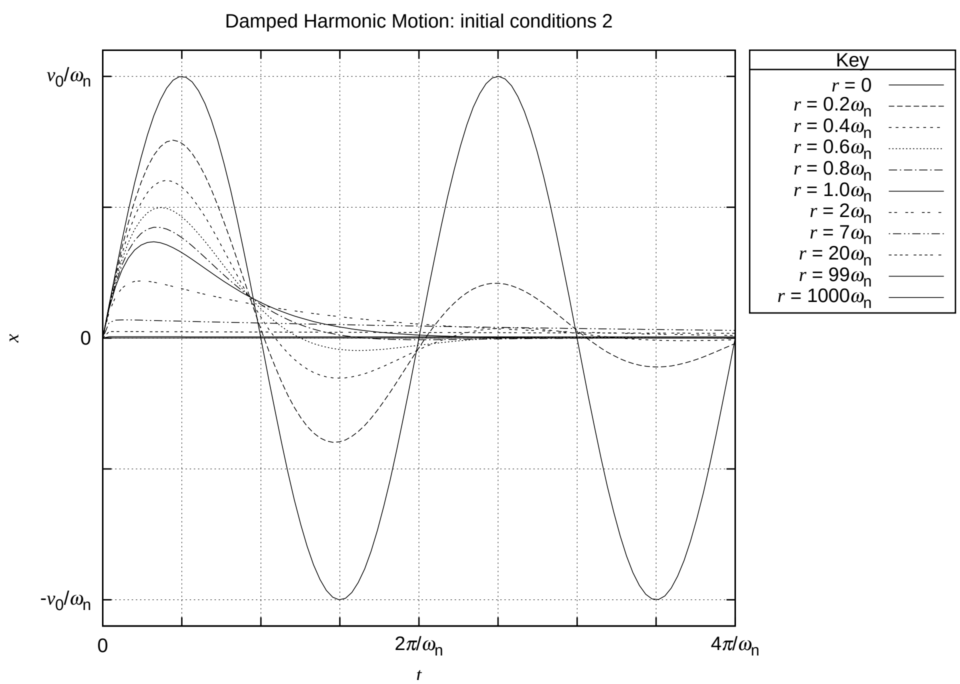

Figure 2: Damped harmonic motion with different levels of damping, with initial conditions 2.

With the second set of initial conditions in Figure 2, where the initial position is the equilibrium position and the initial velocity is v_0, some other facts become more obvious. We see from the highly damped cases that energy is taken out of the system mostly when the object is at high velocity. So we've found that the damping acts in two qualitative ways: 1) it slows the system down (takes out kinetic energy) and 2) it acts more strongly when the system is at higher velocity. Modelling the damping force as negatively proportional to the velocity would take care of both of these qualities (and just so happens to reproduce these graphs exactly, as we'll find out), since the magnitude of the force would be proportional to speed and the work it does would always be negative. Slowing the system down also causes the rate of change of potential energy to slow as well, since the potential energy is proportional to the square of the displacement. After we've done the mathematics to find the formulae for these curves, we'll be able to plot the energy to see exactly what's going on in all of these scenarios.

To finish up with Figure 2 we should just look at the extreme case of infinite damping. Essentially, that's equivalent to smacking a solid system. The system's component parts will not move with respect to each other, so the mass will not move with respect to the damping apparatus (and the spring will just stay unstressed). That produces a flat position-versus-time curve at $x = 0$. Where'd the energy of your smack go? Maybe it went into hurting or deforming the smacker (your hand?), or maybe it went into heat in the damper and noise due to body vibrations of the object itself, or maybe you just smacked your whole object-spring-damper system off of your lab table and onto the floor!

Equation of Motion and Solutions

Now let's go from qualitative to quantitative. Given that an object of mass m has a Hooke restoring force, which is negatively proportional to displacement x from an equilibrium position, and a damping force that is negatively proportional to the object's velocity, the equation of motion is, in three different notations,

$$F_{\mathrm{net},x} = \boxed{-kx - bv_x = ma_x,}$$

$$F_{\mathrm{net},x} = -kx - b\,\tfrac{\mathrm{d}x}{\mathrm{d}t} = m\,\tfrac{\mathrm{d}^2x}{\mathrm{d}t^2}\,,$$

$$F_{\mathrm{net},x} = -kx - b\dot{x} = m\ddot{x}.$$

Rearranging, we have

$$\ddot{x} + 2\left(\tfrac{b}{2m}\right)\dot{x} + \left(\tfrac{k}{m}\right)x = 0.$$

I've multiplied and divided the second term by two because we'll find that $b/2m$ is an important constant in describing the resulting motion. Let's call it r for now: $r = b/2m$. We already know that k/m is the square of the angular frequency of oscillation for an undamped ($b = 0$) oscillator. We'll find that this system exhibits a different frequency of motion than $\sqrt{k/m}$, so we will give a special name to the undamped frequency to distinguish it from the damped oscillation frequency. We'll call it the "natural frequency": $\omega_\mathrm{n} = \sqrt{k/m}$. This name is a little strange, since the absence of damping is a bit unnatural, but this is the common name anyway. So the equation can be written this way:

$$\boxed{\ddot{x} + 2r\dot{x} + \omega_\mathrm{n}^2 x = 0.}$$

The general solutions[1] to this equation come in three different forms depending on how r and ω_n relate to each other. Here is one way to express these general solutions:

Underdamped (light damping)

$$r < \omega_\mathrm{n} \qquad x(t) = A_0 \exp(-rt)\cos\!\left(\sqrt{\omega_\mathrm{n}^2 - r^2}\,t + \phi_0\right) \tag{1}$$

Critically damped

$$r = \omega_\mathrm{n} \qquad x(t) = (Bt + x_0)\exp(-rt) \tag{2}$$

Overdamped (heavy damping)

$$r > \omega_\mathrm{n} \qquad x(t) = C_+ \exp\left[\left(-r + \sqrt{r^2 - \omega_\mathrm{n}^2}\right)t\right] + C_- \exp\left[\left(-r - \sqrt{r^2 - \omega_\mathrm{n}^2}\right)t\right] \tag{3}$$

$$= \exp(-rt)\left[D_\mathrm{c}\cosh\left(\sqrt{r^2 - \omega_\mathrm{n}^2}\,t\right) + D_\mathrm{s}\sinh\left(\sqrt{r^2 - \omega_\mathrm{n}^2}\,t\right)\right] \tag{4}$$

[1] To see a fairly good explanation of the derivation of these solutions, and to see some very interesting information about (non-linear) pendulum motion, see <http://www.chaos.gwdg.de/applets/pendulum/harmosc.htm>.

Note that each of the three cases has in its solution two free parameters (A_0 and ϕ_0, or x_0 and B, or C_+ and C_-). If the initial conditions, that is the initial position and velocity of the oscillator, and the values of r and ω_n are specified, then the values for these parameters will be set. Note also that if you leave the initial conditions and ω_n the same but change r, the parameters will change their values! We'll see that explicitly in a moment.

It will be convenient for this discussion and for memorization to define a couple more variables:

$$\omega_\mathrm{d} = \sqrt{\omega_\mathrm{n}^2 - r^2}$$

$$\beta = \sqrt{r^2 - \omega_\mathrm{n}^2}$$

We'll discuss their meaning and get around to naming r, ω_d, and β below.

If we enforce the initial conditions stipulated in the previous section and take ω_n and r to be given, we find the solutions take the forms given in Table 1 below.

Damping Condition	Solution with Initial Conditions 1 $x(0) = x_0,\ v_x(0) = 0$	Solution with Initial Conditions 2 $x(0) = 0,\ v_x(0) = v_0$
Under. $r < \omega_\mathrm{n}$	$x(t) = x_0 \frac{\omega_\mathrm{n}}{\omega_\mathrm{d}} e^{-rt} \cos\big[\omega_\mathrm{d} t - \arctan(r/\omega_\mathrm{d})\big]$	$x(t) = \frac{v_0}{\omega_\mathrm{d}} e^{-rt} \sin(\omega_\mathrm{d} t)$
Crit. $r = \omega_\mathrm{n}$	$x(t) = x_0 (rt + 1) e^{-rt}$	$x(t) = (v_0 t) e^{-rt}$
Over. $r = \omega_\mathrm{n}$	$x(t) = x_0 \left\{ \frac{1}{2}(1 + \frac{r}{\beta}) e^{(-r+\beta)t} + \frac{1}{2}(1 - \frac{r}{\beta}) e^{(-r-\beta)t} \right\}$ $= x_0\, e^{-rt} \left\{ \cosh(\beta t) + \frac{r}{\beta} \sinh(\beta t) \right\}$	$x(t) = \frac{v_0}{\beta} \left\{ \frac{1}{2} e^{(-r+\beta)t} - \frac{1}{2} e^{(-r-\beta)t} \right\}$ $= \frac{v_0}{\beta} e^{-rt} \sinh(\beta t)$

Table 1: Solutions that conform to the two sets of initial conditions. These are the equations for the graphs in Figures 1 and 2.

Quantitative Analysis of Motion

I'll now discuss each of the three cases of solutions and relate them to the two types of motion observed earlier, and you should refer to Equations 1–4, Table 1, and Figures 1 and 2 above. The solutions should confirm and even build up our intuition about damped harmonic motion.

For the underdamped case, we can first note that the sub-case of $r = 0$ (when $b = 0$) is simply the case without damping and yields simple harmonic motion with natural frequency ω_n. We see that for $r \neq 0$, the motion is an exponentially decaying oscillation, where r is the exponential decay rate of the amplitude for the oscillations. We also see that the frequency of the oscillations, or the frequency of the cyclic part of the solution, has changed from the natural frequency to a lower value. We'll call this new frequency the (under)damped angular frequency $\omega_\mathrm{d} = \sqrt{\omega_\mathrm{n}^2 - r^2}$, where $\omega_\mathrm{d} < \omega_\mathrm{n}$. [2] This means that when damping is increased, the oscillation becomes slower, with a longer period. As we approach the critical case (as r increases to almost reach ω_n), ω_d approaches zero and the period of the oscillation approaches infinity, just as we observed earlier.

If we look at the solutions for initial conditions 1 in Table 1, we can see that as r goes to ω_n, the factor $x_0 \frac{\omega_\mathrm{n}}{\omega_\mathrm{d}} \cos\big[\omega_\mathrm{d} t - \arctan(r/\omega_\mathrm{d})\big]$ in the first solution becomes $x_0(rt + 1)$ in the second solution. The cosine

[2] Don't confuse the (under)damping angular frequency ω_d with a frequency encountered in sinusoidally forced harmonic motion, the "forcing angular frequency" ω_f, which is often called the "driving frequency".

gets stretched horizontally (with a longer period) and vertically (with a larger amplitude) and shifted (where the shift approaches $-\pi/2$, nearing the unshifted sine) in such a way that it turns into a straight line intersecting $x = x_0$ with a slope of $x_0 r$. The slope-factor r competes with the exponential decay (which initially has a slope of $-r$) to allow the resulting function to start off with zero slope and thus zero velocity and satisfy the initial conditions. (The product rule for derivatives dictates that the competing slopes are weighted by the values of the other function: given $f = gh$, $\dot{f} = h\dot{g} + g\dot{h}$, and in our case $h = 1$ and $g = 1$ at $t = 0$.) With this effect the curve looks more and more like an exponential decay with some initial hints of a cosine at the beginning. Note that with these solutions we can calculate exactly how the first intersection of the curve of motion with the time axis shoots off to infinity as we approach the critical case. Of course, it is much easier to see how, with the second set of solutions in Table 1, the factor $\frac{v_0}{\omega_\mathrm{d}} \sin(\omega_\mathrm{d} t)$ becomes $(v_0 t)$ as r goes to ω_n.

For the critically damped case, the motion is not oscillatory and neither is it purely exponential, unless B happens to be zero. (Neither of our initial conditions cause B to be zero, but B could be zero under different conditions, where the initial velocity is negative.) The parameter r still gives the decay rate for the exponential part of the solution, but the overall value of x has two time-dependent factors: $(Bt + x_0)$ and $\exp(-rt)$. After enough time, however, the exponential part of the solution will be dominant, meaning that eventually it will be hard to tell the difference between the $B = 0$ case and the $B \neq 0$ case, since the exponential $\exp(-rt)$ eventually shrinks a lot faster than t grows. So, although this solution isn't necessarily purely exponential decay, you could say that in the long run (or asymptotically) it is exponential.

For the overdamped case, we see two competing pure exponentially decaying terms; both exponents have negative decay rates. However, the result of adding these terms can sometimes look (for a short while) distinctly non-exponential. (See, for example, the overdamped "bumps" in Figure 2.) In the argument for each exponential term we see something that looks like ω_d but is different; we've given it the symbol β: $\beta = \sqrt{r^2 - \omega_\mathrm{n}^2}$. Note that $\beta < r$ (that's why the decay rates are both negative), and β is closer to r for larger values of r. We could call β the "decay rates' offset" (from r) or perhaps the "hyperbolic phase velocity".

We can also now name r. For under- and critical damping, r can definitely be called the "exponential decay rate", but for overdamping it's a little more complicated and harder to come up with a name that encompasses it's function in all the ways you can express the solution. Maybe if we call it "*an* exponential decay rate" rather than "*the* exponential decay rate", that would be satisfactory.

I'll do a quick analysis now and clean this up later. (Explain how the critically damped case is equivalent to the overdamped case when r is slightly greater than ω_n. Explain how the two terms allow there to be a hint of a cosine with the first initial conditions.) It appears that one of the exponential terms does the job of either easing the object into a long exponential decay (first initial conditions) or getting rid of the kinetic energy and then easing it into a long exponential decay (second initial conditions). As r goes to infinity, one of the decays becomes incredibly fast and one becomes incredibly slow. At the extreme of infinite damping, the infinitely-fast decay immediately rids the system of kinetic energy and the infinitely-slow decay (no decay) holds the object at it's initial position.

Energy of Damped Harmonic Motion

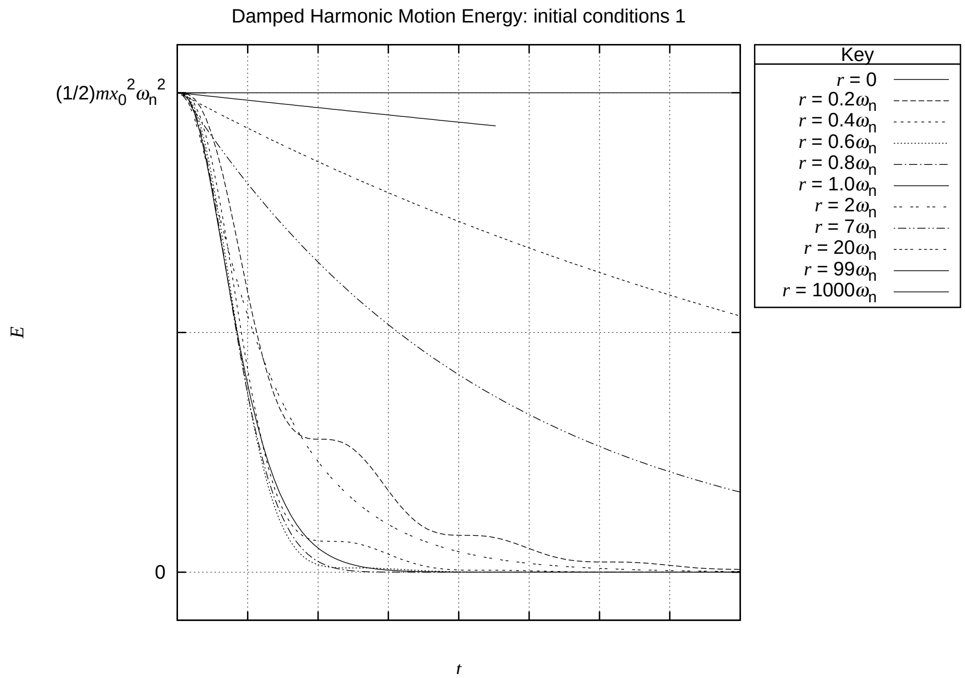

Figure 3: Energy of harmonic motion with different levels of damping, with initial conditions 1.

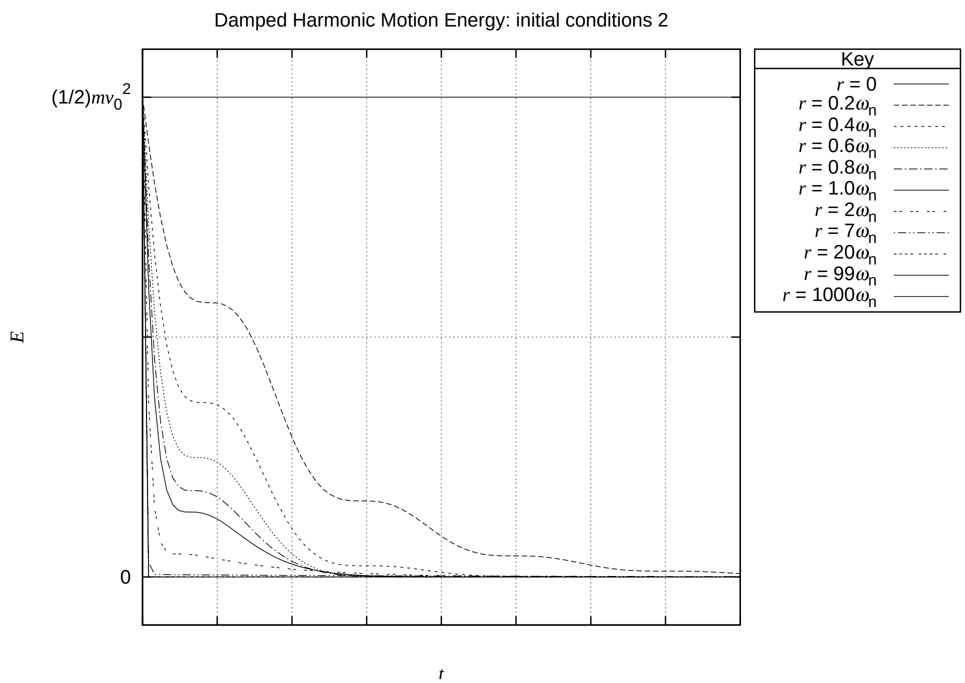

Figure 4: Energy of harmonic motion with different levels of damping, with initial conditions 2.

Doors, Shocks

Search online for "critically damped" or "critical damping" and "shocks", and you can find some interesting articles, such as the one I found here: `<http://www.modified.com/editors/0708_sccp_jay_chen_editorial/index.html>`

Summary of Equation and Solutions

$$\frac{\partial^2 x}{\partial t^2} + 2r\frac{\partial x}{\partial t} + \omega_\mathrm{n}^2 x = 0.$$

Underdamped (light damping)		
	$r < \omega_\mathrm{n}$	$x(t) = A(t)\cos(\omega_\mathrm{d} t + \phi_0)$
Critically damped		
	$r = \omega_\mathrm{n}$	$x(t) = (Bt + x_0)\, e^{-rt}$
Overdamped (heavy damping)		
	$r > \omega_\mathrm{n}$	$x(t) = C_+ e^{(-r+\beta)t} + C_- e^{(-r-\beta)t}$
		$= e^{-rt}\left[D_\mathrm{c}\cosh(\beta t) + D_\mathrm{s}\sinh(\beta t)\right]$

- $A(t)$ time-dependent (exponentially decaying) amplitude
  - $A(t) = A_0 e^{-rt}$
- r an exponential decay rate .... damped exponential phase velocity
  - $r = \frac{\text{(damping constant)}}{2\times\text{(inertial constant)}}$
  - Examples: $\frac{b}{2m}$, $\frac{\Gamma}{2I}$, etc.
- ω_n natural angular frequency .... undamped angular phase velocity
  - $\omega_\mathrm{n} = \sqrt{\frac{\text{(restoring constant)}}{\text{(inertial constant)}}}$
  - Examples: $\sqrt{\frac{k}{m}}$, $\sqrt{\frac{\kappa}{I}}$, etc.
- ω_d (under)damped angular frequency .... (under)damped angular phase velocity
  - $\omega_\mathrm{d} = \sqrt{\omega_\mathrm{n}^2 - r^2}$
- β hyperbolic phase velocity .... (over)damped hyperbolic phase velocity
  - $\beta = \sqrt{r^2 - \omega_\mathrm{n}^2}$