Gravito-Electromagnetism (GEM)

Weak (Linearizable) Slowly Changing Gravitation

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Contents

1 Introduction

Instead of imagining space-time as being warped by mass and energy, one can speak of a classical spin-2 graviton field (in flat, Minkowski space-time) that generates gravitation. Although we don't know yet how to quantize this field, we can think of it in a way similar to how we think of electromagnetism being mediated by photons. And just as a $1/r^2$ Coulomb force generates magnetism when the finite speed of the mediating photon is taken into account[1], a $1/r^2$ Newtonian gravitational force generates "gravito-magnetism" when the finite speed of the mediating graviton is taken into account. Magnetism is fundamentally an electric-force effect[2], and gravity must have some analogous "magnetic" force, meaning a gravitational force proportional and perpendicular to the velocity of a test mass. Einstein showed that gravity should be non-linear, so we know that the graviton should self-interact. (General relativity also implies that the graviton should be spin 2.) Taking that self-interaction (and spin-2) into account should bring us all the way to the equivalent of general relativity. But it may be that in most of the

[1] See Chris Clark's paper "Magnetism is not fundamental" at `<http://dfcd.net/articles/magnetism.pdf>`.

[2] Even the quantum spin magnetic moment is seen to be a natural electric effect in quantum field theory; see "What is Spin?" by Hans C. Ohanian, available on my website at `<http://aforrester.bol.ucla.edu/educate.php#Outsource>`.

universe (barring black holes, supernovae, et cetera), all you really need to know about gravitation is the electromagnetic-analogue part (gravito-electromagnetism or GEM) to have an accurate description.

In this paper, instead of taking the "bottom-up" approach just discussed (in which one constructs GEM by starting from the Newtonian gravity and taking the finite speed of the graviton into account), we take the "top-down" approach and start with the equations of general relativity, apply some simplifying assumptions, and obtain the GEM equations. After doing some preparation in sections 2 – 5, we show that under two assumptions,

(1) low mass-energy density or weak gravitational fields (so we may linearize the Einstein equations)

(2) slow changes[3] in matter-energy (so we may neglect second order time derivatives)

Einstein's field equations of general relativity,

$$G_{\mu\nu} = \frac{8\pi G_{\mathrm{N}}}{c^4} T_{\mu\nu},$$

become

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{G} &\simeq -\tfrac{1}{\varepsilon_{\mathrm{g}}}\rho^{(\mathrm{e})\mathrm{f}} - \tfrac{1}{\varepsilon_{\mathrm{g}}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} && (1)\\ \tfrac{1}{4c}\boldsymbol{\nabla}\times\mathbf{H} - \tfrac{1}{c^2}\partial_t\mathbf{G} &\simeq -\mu_{\mathrm{g}}\mathbf{J}^{(\mathrm{e})\mathrm{f}} + \mu_{\mathrm{g}}\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}} + \mu_{\mathrm{g}}\partial_t\boldsymbol{\mathcal{P}}, && (2)\end{aligned}$$

very much like the electromagnetic field equations in matter,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \tfrac{1}{\varepsilon_0}\rho^{\mathrm{f}} - \tfrac{1}{\varepsilon_0}\boldsymbol{\nabla}\cdot\mathbf{P} && (3)\\ \boldsymbol{\nabla}\times\mathbf{B} - \tfrac{1}{c^2}\partial_t\mathbf{E} &= \mu_0\mathbf{J}^{\mathrm{f}} + \mu_0\boldsymbol{\nabla}\times\mathbf{M} + \mu_0\partial_t\mathbf{P}. && (4)\end{aligned}$$

We also discuss the meaning of the differences between these equations. In the preparation work, we linearize all the tensors and the Christoffel connection symbols, which takes care of assumption 1 above, by taking the metric to be the Minkowski metric plus a perturbation. In section 6 we apply the second assumption.

Furthermore, in section 7 we show that the geodesic equation for particle trajectories in space-time,

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0,$$

becomes the "gravito-Lorentz" force,

$$\tfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{gL}} = E\left[\mathbf{G} + \left(\tfrac{\mathbf{v}}{c}\times\mathbf{H}\right)\right],$$

analogous to the electromagnetic Lorentz force:

$$\tfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{L}} = Q\left[\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right].$$

And finally, in section 8, we do an example problem dealing with the Lense-Thirring effect, which relates to "inertial frame dragging" in general relativity.

2 Notation

We use [4]

$$\partial^2 = \partial_\mu\partial^\mu = -{\partial_0}^2 + {\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2 = -\tfrac{1}{c^2}{\partial_t}^2 + {\partial_x}^2 + {\partial_y}^2 + {\partial_z}^2 = -{\partial_0}^2 + \nabla^2$$

[3]To be more accurate, we should use a word like "acceleration" rather than change, since "slow change" implies small first derivative with respect to time.

[4]Sometimes ∂^2 is written as $\Box$, in analogy to Δ, and sometimes it's written as $\Box^2$, in analogy to ∇^2, usually in physics settings. I prefer the $\Box^2$ notation because it reveals its squared nature (while the box itself reveals its four-variable nature).

$$K_\mathrm{e} \equiv \frac{1}{4\pi\varepsilon_0} \qquad \text{and} \qquad K_\mathrm{m} \equiv \frac{\mu_0}{4\pi}$$

We use $\bar{h}$ to denote some form of contraction of the tensor h, for example, taking the "pseudo-trace" of $h_{\mu\nu}$: $\bar{h} = h^\mu{}_\mu = g^{\mu\rho}h_{\rho\mu}$; or the following contraction: $\bar{h}_\beta{}^\gamma{}_\epsilon = h^\alpha{}_\beta{}^\gamma{}_{\alpha\epsilon} = g_{\alpha\delta}h^\alpha{}_\beta{}^{\gamma\delta}{}_\epsilon$. We will thus put bars over the Riemann tensor symbol (R) to denote the Ricci tensor ($\bar{R}$) and Ricci scalar ($\bar{\bar{R}}$) and over the Einstein tensor symbol (G) to denote the Einstein scalar ($\bar{G}$), showing that they are contractions. We will use G_N for Newton's gravitational constant (except when accompanied by a mass M):

$$\bar{G} = G^\mu{}_\mu \neq G \neq G_\mathrm{N}$$

$$GM \equiv G_\mathrm{N}M$$

Also, we will use $\mathbf{G}$ and G^i to denote the gravito-electric field. With these notations, the only ambiguity is between the Einstein tensor G (when writing it without super- or subscripts) and the magnitude of the gravito-electric field $|\mathbf{G}| = G$ (which I think does not turn up in this document), so long as they are not multiplied by a mass M.

3 The Metric, Perturbation, and Potentials

We may describe the metric g as the Minkowski metric $\eta = \mathrm{diag}(-1,1,1,1)$ plus a perturbation h, where $\left|h_{\mu\nu}\right| \ll 1$ for some appropriately chosen coordinate system:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The inverse is

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$$

since

$$\begin{aligned} g_{\mu\nu}g^{\nu\rho} &\simeq (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\rho} - h^{\nu\rho}) \\ &= \delta^\rho_\mu + \cancel{h_{\mu\nu}\eta^{\nu\rho}} - \cancel{\eta_{\mu\nu}h^{\nu\rho}} + h_{\mu\nu}h^{\nu\rho} \\ &= \delta^\rho_\mu + \xcancel{O(h^2)} \end{aligned}$$

We will decompose h into components that later be seen as analogous to the electric scalar potential and magnetic vector potential under the appropriate circumstances. This "potentials decomposition" of h is the following:

$$\begin{aligned} h_{\mu\nu} &= \begin{pmatrix} -2\Phi & w_1 & w_2 & w_3 \\ w_1 & 2(s_{11} - \Psi) & 2s_{12} & 2s_{13} \\ w_2 & 2s_{21} & 2(s_{22} - \Psi) & 2s_{23} \\ w_3 & 2s_{31} & 2s_{32} & 2(s_{33} - \Psi) \end{pmatrix}_{\mu\nu} \\ &= \begin{pmatrix} -2\Phi & w_j \\ w^i & 2(s_{ij} - \Psi\delta_{ij}) \end{pmatrix}_{\mu\nu} = \begin{pmatrix} -2\Phi & w_j \\ w^i & h_{ij} \end{pmatrix}_{\mu\nu} \end{aligned}$$

where h and s are symmetric since g and η are symmetric, and

$$\begin{aligned} \Psi &\equiv -\tfrac{1}{6}h^i{}_i \\ s_{ij} &\equiv \tfrac{1}{2}\left(h_{ij} - \tfrac{1}{3}h^k{}_k\delta_{ij}\right) \\ &= \tfrac{1}{2}h_{ij} + \Psi\delta_{ij} \end{aligned}$$

so that s is traceless: $\bar{s} = s_{11} + s_{22} + s_{33} = \delta^{ij} s_{ij} = \frac{1}{2}(\delta^{ij} h_{ij} - \frac{1}{3}\delta^{ij} h^k{}_k \delta_{ij}) = \frac{1}{2}(h^j{}_j - h^k{}_k) = 0$. Note that $h^i{}_i = \delta^{ij} h_{ji} = -6\Psi \neq \bar{h} = h^\mu{}_\mu \simeq \eta^{\mu\nu} h_{\nu\mu}$ and that $\bar{h}$ cannot be called the trace of h, although it might be called the pseudo-trace of h. We have

$$\begin{aligned} \bar{h} &= h^\mu{}_\mu \\ &= g^{\mu\nu} h_{\nu\mu} \\ &\simeq \eta^{\mu\nu} h_{\nu\mu} \qquad \text{(equality to first order in } h) \\ &= -h_{00} + h_{11} + h_{22} + h_{33} \\ &= 2\Phi + 2(s_{11} - \Psi) + 2(s_{22} - \Psi) + 2(s_{33} - \Psi) \\ &= 2(\Phi + \not{\bar{s}} - 3\Psi) \\ &= 2(\Phi - 3\Psi) \end{aligned}$$

Also, for reference, we have $h_{00} = -2\Phi$, $h_{0i} = w_i$, and $h_{ij} = 2(s_{ij} - \Psi\delta_{ij})$.

3.1 Dimensional Analysis

Note that $g = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$. While the metric tensor $g_{\mu\nu}$ is a dimensionless type-(0,2) tensor field, the metric g is an area-valued 2-form field. [5]

3.2 Notable Metrics

The general, linearized metric is

$$g = -(1 + 2\Phi)c^2 \mathrm{d}t^2 + 2w_i \, \mathrm{d}t \, \mathrm{d}x^i + \big[(1 - 2\Psi)\delta_{ij} + 2s_{ij}\big] \mathrm{d}x^i \mathrm{d}x^j$$

Given that $\Psi = \Phi$ and $s = 0$,

$$g = -(1 + 2\Phi)c^2 \mathrm{d}t^2 + (1 - 2\Phi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

The Schwarzschild metric is

$$g = -\left(1 - \tfrac{2GM}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \tfrac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

[5] Are both g and $g_{\mu\nu}$ called "the metric tensor"? If so, what would be a good way to verbally distinguish them?

4 The Connection and Tensors of Interest

In terms of the perturbation h, and to first order in h, we have that the Christoffel connection Γ, Riemann tensor R, Ricci tensor $\bar{R}$, Ricci scalar $\bar{\bar{R}}$, and Einstein tensor G are

$$\begin{aligned}
\Gamma^{\rho}_{\mu\nu} &= \tfrac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\
&\simeq \tfrac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\
R^{\rho}{}_{\mu\sigma\nu} &= \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \cancel{\Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu}} - \cancel{\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}} \\
&\simeq \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} \\
&\simeq \partial_\sigma\left(\tfrac{1}{2}\eta^{\rho\lambda}(\cancel{\partial_\nu h_{\mu\lambda}} + \partial_\mu h_{\lambda\nu} - \partial_\lambda h_{\nu\mu})\right) - \partial_\nu\left(\tfrac{1}{2}\eta^{\rho\lambda}(\cancel{\partial_\sigma h_{\mu\lambda}} + \partial_\mu h_{\lambda\sigma} - \partial_\lambda h_{\sigma\mu})\right) \\
&= \tfrac{1}{2}\eta^{\rho\lambda}\left(\partial_\sigma\partial_\mu h_{\lambda\nu} - \partial_\sigma\partial_\lambda h_{\nu\mu} - \partial_\nu\partial_\mu h_{\lambda\sigma} + \partial_\nu\partial_\lambda h_{\sigma\mu}\right) \\
&= \tfrac{1}{2}\left(\partial_\sigma\partial_\mu h^{\rho}{}_{\nu} - \partial_\sigma\partial^\rho h_{\nu\mu} - \partial_\nu\partial_\mu h^{\rho}{}_{\sigma} + \partial_\nu\partial^\rho h_{\sigma\mu}\right) \\
\bar{R}_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu} \\
&\simeq \tfrac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} - \partial_\rho\partial^\rho h_{\nu\mu} - \partial_\nu\partial_\mu h^{\rho}{}_{\rho} + \partial_\nu\partial^\rho h_{\rho\mu}\right) \\
&= \tfrac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} - \partial^2 h_{\nu\mu} - \partial_\nu\partial_\mu \bar{h} + \partial_\nu\partial_\rho h^{\rho}{}_{\mu}\right) \\
&= \tfrac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu \bar{h} - \partial^2 h_{\mu\nu}\right) \qquad \text{(rearranging and using symmetry of } h_{\mu\nu}\text{)} \\
\bar{\bar{R}} &= R^{\nu}{}_{\nu} \\
&\simeq \tfrac{1}{2}\left(\partial_\rho\partial^\nu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho\nu} - \partial_\nu\partial^\nu \bar{h} - \partial^2 h_{\nu}{}^{\nu}\right) \\
&= \tfrac{1}{2}\left(\partial_\rho\partial^\nu h^{\rho}{}_{\nu} + \partial^\rho\partial_\nu h_{\rho}{}^{\nu} - \partial^2 \bar{h} - \partial^2 \bar{h}\right) \\
&= \partial_\rho\partial^\nu h^{\rho}{}_{\nu} - \partial^2 \bar{h} \\
&= \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 \bar{h} \qquad \text{(rearranging and renaming indices)} \\
G_{\mu\nu} &= \bar{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\bar{\bar{R}} \\
&\simeq \bar{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\bar{\bar{R}} \\
&= \tfrac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu \bar{h} - \partial^2 h_{\mu\nu}\right) - \tfrac{1}{2}\eta_{\mu\nu}\left(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 \bar{h}\right) \\
&= \tfrac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}{}_{\nu} + \partial_\rho\partial_\nu h^{\rho}{}_{\mu} - \partial_\mu\partial_\nu \bar{h} - \partial^2 h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\partial^2 \bar{h}\right)
\end{aligned}$$

We could also derive this linearized Einstein tensor by varying the following Lagrangian with respect to the perturbation $h_{\mu\nu}$:

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu \bar{h}) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^{\mu}{}_{\sigma}) + \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu \bar{h})(\partial_\nu \bar{h})\right]$$

In terms of the “potentials decomposition” of h, we have

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &\simeq \tfrac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\
\Gamma^0_{00} &\simeq \tfrac{1}{2}\eta^{00}(\partial_0 h_{00} + \cancel{\partial_0 h_{00}} - \cancel{\partial_0 h_{00}}) \\
&= \partial_0\Phi \\
\Gamma^i_{00} &\simeq \tfrac{1}{2}\eta^{ii}(\partial_0 h_{0i} + \partial_0 h_{i0} - \partial_i h_{00}) \\
&= \partial_i\Phi + \partial_0 h_{0i} \\
&= \partial_i\Phi + \partial_0 w_i \\
\Gamma^0_{j0} &\simeq \tfrac{1}{2}\eta^{00}(\partial_j h_{00} + \cancel{\partial_0 h_{0j}} - \cancel{\partial_0 h_{j0}}) \\
&= \partial_j\Phi \\
\Gamma^i_{j0} &\simeq \tfrac{1}{2}\eta^{ii}(\partial_j h_{0i} + \partial_0 h_{ij} - \partial_i h_{j0}) \\
&= \tfrac{1}{2}(\partial_j w_i + \partial_0 h_{ij} - \partial_i w_j) \\
&= \partial_{[j}w_{i]} + \tfrac{1}{2}\partial_0 h_{ij} \\
\Gamma^0_{jk} &\simeq \tfrac{1}{2}\eta^{00}(\partial_j h_{k0} + \partial_k h_{0j} - \partial_0 h_{jk}) \\
&= -\tfrac{1}{2}(\partial_j w_k + \partial_k w_j - \partial_0 h_{jk}) \\
&= -\partial_{(j}w_{k)} + \tfrac{1}{2}\partial_0 h_{jk} \\
\Gamma^i_{jk} &\simeq \tfrac{1}{2}\eta^{ii}(\partial_j h_{ki} + \partial_k h_{ij} - \partial_i h_{jk}) \\
&= \partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}
\end{aligned}$$

$$\begin{aligned}
R_{\tau\mu\sigma\nu} &= g_{\tau\rho}R^\rho{}_{\mu\sigma\nu} \\
&\simeq (\eta_{\tau\rho} + \cancel{h_{\tau\rho}})(\partial_\sigma\Gamma^\rho_{\nu\mu} - \partial_\nu\Gamma^\rho_{\sigma\mu}) \\
&= \eta_{\tau\rho}(\partial_\sigma\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\sigma}) \\
R_{0j0l} &\simeq \eta_{00}(\partial_0\Gamma^0_{jl} - \partial_l\Gamma^0_{j0}) \\
&= -\left(\partial_0\left(-\partial_{(j}w_{l)} + \tfrac{1}{2}\partial_0 h_{jl}\right) - \partial_l(\partial_j\Phi)\right) \\
&= \partial_j\partial_l\Phi + \partial_0\partial_{(j}w_{l)} - \tfrac{1}{2}{\partial_0}^2 h_{jl} \\
R_{0jkl} &\simeq \eta_{00}(\partial_k\Gamma^0_{jl} - \partial_l\Gamma^0_{jk}) \\
&= -\left(\partial_k\left(-\partial_{(j}w_{l)} + \tfrac{1}{2}\partial_0 h_{jl}\right) - \partial_l\left(-\partial_{(j}w_{k)} + \tfrac{1}{2}\partial_0 h_{jk}\right)\right) \\
&= \partial_k\partial_{(j}w_{l)} - \tfrac{1}{2}\partial_k\partial_0 h_{jl} - \partial_l\partial_{(j}w_{k)} + \tfrac{1}{2}\partial_l\partial_0 h_{jk} \\
&= \tfrac{1}{2}\partial_k(\partial_j w_l + \cancel{\partial_l w_j}) - \tfrac{1}{2}\partial_l(\partial_j w_k + \cancel{\partial_k w_j}) + \tfrac{1}{2}(\partial_l\partial_0 h_{jk} - \partial_k\partial_0 h_{jl}) \\
&= \tfrac{1}{2}(\partial_k\partial_j w_l - \partial_l\partial_j w_k) + \partial_0\tfrac{1}{2}(\partial_l h_{kj} - \partial_k h_{lj}) \\
&= \partial_j\tfrac{1}{2}(\partial_k w_l - \partial_l w_k) + \partial_0\partial_{[l}h_{k]j} \\
&= \partial_j\partial_{[k}w_{l]} + \partial_0\partial_{[l}h_{k]j} \\
R_{ijkl} &\simeq \eta_{ii}(\partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk}) \\
&= \partial_k(\partial_{(j}h_{l)i} - \tfrac{1}{2}\partial_i h_{jl}) - \partial_l(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}) \\
&= \partial_k\partial_{(j}h_{l)i} - \tfrac{1}{2}\partial_k\partial_i h_{jl} - \partial_l\partial_{(j}h_{k)i} + \tfrac{1}{2}\partial_l\partial_i h_{jk} \\
&= \partial_k\tfrac{1}{2}(\partial_j h_{li} + \cancel{\partial_l h_{ji}}) - \partial_l\tfrac{1}{2}(\partial_j h_{ki} + \cancel{\partial_k h_{ji}}) + \tfrac{1}{2}(\partial_l\partial_i h_{jk} - \partial_k\partial_i h_{jl}) \\
&= \partial_j\tfrac{1}{2}(\partial_k h_{li} - \partial_l h_{ki}) + \partial_i\tfrac{1}{2}(\partial_l h_{kj} - \partial_k h_{lj}) \\
&= \partial_j\partial_{[k}h_{l]i} + \partial_i\partial_{[l}h_{k]j}
\end{aligned}$$

$$
\begin{aligned}
\bar{R}_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu} \\
\bar{R}_{00} &= R^{\rho}{}_{0\rho 0} \\
&= R^{0}{}_{000} + R^{1}{}_{010} + R^{2}{}_{020} + R^{3}{}_{030} \\
&\simeq -R_{0000} + R_{1010} + R_{2020} + R_{3030} \qquad \text{(lowering indices with } \eta\text{, thereby keeping only terms first order in } h\text{)} \\
&= \cancel{R_{0000}} + R_{0101} + R_{0202} + R_{0303} \qquad \text{(by antisymmetry in last pair and first pair)} \\
&\simeq \left(\partial_1\partial_1\Phi + \partial_0\partial_{(1}w_{1)} - \tfrac{1}{2}\partial_0\partial_0 h_{11}\right) + \left(\partial_2\partial_2\Phi + \partial_0\partial_{(2}w_{2)} - \tfrac{1}{2}\partial_0\partial_0 h_{22}\right) + \left(\partial_3\partial_3\Phi + \partial_0\partial_{(3}w_{3)} - \tfrac{1}{2}\partial_0\partial_0 h_{33}\right) \\
&= \nabla^2\Phi + \partial_0\partial_1 w_1 + \partial_0\partial_2 w_2 + \partial_0\partial_3 w_3 - \tfrac{1}{2}{\partial_0}^2(h_{11} + h_{22} + h_{33}) \\
&= \nabla^2\Phi + \partial_0 \boldsymbol{\nabla}\cdot\mathbf{w} - {\partial_0}^2\left((s_{11} - \Psi) + (s_{22} - \Psi) + (s_{33} - \Psi)\right) \\
&= \nabla^2\Phi + \partial_0 \boldsymbol{\nabla}\cdot\mathbf{w} - {\partial_0}^2\left(\cancel{s} - 3\Psi\right) \\
&= \nabla^2\Phi + \partial_0 \boldsymbol{\nabla}\cdot\mathbf{w} + 3{\partial_0}^2\Psi
\end{aligned}
$$

$$
\begin{aligned}
\bar{R}_{0j} &= R^{\rho}{}_{0\rho j} \\
&= R^{0}{}_{00j} + R^{1}{}_{01j} + R^{2}{}_{02j} + R^{3}{}_{03j} \\
&\simeq \cancel{R_{000j}} + R_{101j} + R_{202j} + R_{303j} \\
&= -R_{011j} - R_{022j} - R_{033j} \\
&\simeq -\left(\partial_1\partial_{[1}w_{j]} + \partial_0\partial_{[j}h_{1]1}\right) - \left(\partial_2\partial_{[2}w_{j]} + \partial_0\partial_{[j}h_{2]2}\right) - \left(\partial_3\partial_{[3}w_{j]} + \partial_0\partial_{[j}h_{3]3}\right) \\
&= -\left(\partial_1\partial_{[1}w_{j]} + \partial_2\partial_{[2}w_{j]} + \partial_3\partial_{[3}w_{j]}\right) - \partial_0\left(\partial_{[j}h_{1]1} + \partial_{[j}h_{2]2} + \partial_{[j}h_{3]3}\right) \\
&= -\tfrac{1}{2}\left(\partial_1\partial_1 w_j + \partial_2\partial_2 w_j + \partial_3\partial_3 w_j\right) + \tfrac{1}{2}\left(\partial_1\partial_j w_1 + \partial_2\partial_j w_2 + \partial_3\partial_j w_3\right) \\
&\qquad -\tfrac{1}{2}\partial_0\left(\partial_j h_{11} + \partial_j h_{22} + \partial_j h_{33}\right) + \tfrac{1}{2}\partial_0\left(\partial_1 h_{j1} + \partial_2 h_{j2} + \partial_3 h_{j3}\right) \\
&= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} - \tfrac{1}{2}\partial_0(\partial_j h^{k}{}_{k} - \partial_k {h_j}^{k}) \\
&= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} - \tfrac{1}{2}\partial_0\left(-6\partial_j\Psi - 2\partial_k({s_j}^{k} - \delta_j^k\Psi)\right) \\
&= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\left(3\partial_j\Psi + \partial_k {s_j}^{k} - \partial_j\Psi\right) \\
&= -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\partial_k {s_j}^{k} + 2\partial_0\partial_j\Psi
\end{aligned}
$$

$$
\begin{aligned}
\bar{R}_{ij} &= R^{\rho}{}_{i\rho j} \\
&= R^{0}{}_{i0j} + R^{1}{}_{i1j} + R^{2}{}_{i2j} + R^{3}{}_{i3j} \\
&\simeq -R_{0i0j} + R_{1i1j} + R_{2i2j} + R_{3i3j} \\
&\simeq -\left(\partial_i\partial_j\Phi + \partial_0\partial_{(i}w_{j)} - \tfrac{1}{2}{\partial_0}^2 h_{ij}\right) \\
&\qquad + \left(\partial_i\partial_{[1}h_{j]1} + \partial_1\partial_{[j}h_{1]i}\right) + \left(\partial_i\partial_{[2}h_{j]2} + \partial_2\partial_{[j}h_{2]i}\right) + \left(\partial_i\partial_{[3}h_{j]3} + \partial_3\partial_{[j}h_{3]i}\right) \\
&= \left(-\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} + \tfrac{1}{2}{\partial_0}^2 h_{ij}\right) + \partial_i\left(\partial_{[1}h_{j]1} + \partial_{[2}h_{j]2} + \partial_{[3}h_{j]3}\right) + \left(\partial_1\partial_{[j}h_{1]i} + \partial_2\partial_{[j}h_{2]i} + \partial_3\partial_{[j}h_{3]i}\right) \\
&= \left(-\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} + \tfrac{1}{2}{\partial_0}^2 h_{ij}\right) + \partial_i\tfrac{1}{2}(\partial_1 h_{j1} + \partial_2 h_{j2} + \partial_3 h_{j3}) - \partial_i\tfrac{1}{2}(\partial_j h_{11} + \partial_j h_{22} + \partial_j h_{33}) \\
&\qquad + \tfrac{1}{2}(\partial_1\partial_j h_{1i} + \partial_2\partial_j h_{2i} + \partial_3\partial_j h_{3i}) - \tfrac{1}{2}(\partial_1\partial_1 h_{ji} + \partial_2\partial_2 h_{ji} + \partial_3\partial_3 h_{ji}) \\
&= \left(-\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} + \tfrac{1}{2}{\partial_0}^2 h_{ij}\right) + \tfrac{1}{2}\partial_i\partial_k {h_j}^{k} - \tfrac{1}{2}\partial_i\partial_j\delta^{kl}h_{kl} + \tfrac{1}{2}\partial_j\partial_k {h_i}^{k} - \tfrac{1}{2}\nabla^2 h_{ij} \\
&= -\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} - \tfrac{1}{2}\partial^2 h_{ij} + \partial_k\partial_{(i}{h_{j)}}^{k} - \tfrac{1}{2}\partial_i\partial_j\delta^{kl}h_{kl} \\
&= -\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} - \tfrac{1}{2}\partial^2 2(s_{ij} - \Psi\delta_{ij}) + \partial_k 2\left(\partial_{(i}{s_{j)}}^{k} - \partial_{(i}\delta^k_{j)}\Psi\right) - \tfrac{1}{2}\partial_i\partial_j(-6\Psi) \\
&= -\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + \partial^2\Psi\delta_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^{k} - (\partial_k\partial_i\delta^k_j\Psi + \partial_k\partial_j\delta^k_i\Psi) + 3\partial_i\partial_j\Psi \\
&= -\partial_i\partial_j\Phi - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + \partial^2\Psi\delta_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^{k} - 2\partial_i\partial_j\Psi + 3\partial_i\partial_j\Psi \\
&= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^{k} + \partial^2\Psi\delta_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{R}} &= R^\mu{}_\mu \\
&= R^0{}_0 + R^i{}_i \\
&\simeq -R_{00} + R^i{}_i \qquad \text{(to first order in } h\text{)} \\
&\simeq -\left(\nabla^2\Phi + \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} + 3{\partial_0}^2\Psi\right) + \left(-\partial^i\partial_i(\Phi-\Psi) - \partial_0\partial^{(i}w_{i)} - \partial^2\cancel{s^i}_i + 2\partial_k\partial^{(i}s_{i)}{}^k + \partial^2\Psi\delta^i_i\right) \\
&= -\nabla^2\Phi - \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - 3{\partial_0}^2\Psi - \nabla^2(\Phi-\Psi) - \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} + 2\partial_k\partial^i s_i{}^k + 3\partial^2\Psi \\
&= -\nabla^2(2\Phi-\Psi) - 2\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - 3{\partial_0}^2\Psi + 2\partial_k\partial_i s^{ik} - 3({\partial_0}^2 - \nabla^2)\Psi \\
&= -2\nabla^2(\Phi-2\Psi) - 2\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} + 2\partial_k\partial_i s^{ik} - 6{\partial_0}^2\Psi \\
&= 2\left(-\nabla^2(\Phi-2\Psi) - \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_i\partial_j s^{ij} - 3{\partial_0}^2\Psi\right)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{\mu\nu} &= \bar{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\bar{\bar{R}} \\
&\simeq \bar{R}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\bar{\bar{R}} \\
G_{00} &\simeq \bar{R}_{00} - \tfrac{1}{2}\eta_{00}\bar{\bar{R}} \\
&\simeq \left(\cancel{\nabla^2\Phi} + \cancel{\partial_0\boldsymbol{\nabla}\cdot\mathbf{w}} + \cancel{3{\partial_0}^2\Psi}\right) + \tfrac{1}{2}2\left(-\nabla^2(\cancel{\Phi}-2\Psi) - \cancel{\partial_0\boldsymbol{\nabla}\cdot\mathbf{w}} + \partial_i\partial_j s^{ij} - \cancel{3{\partial_0}^2\Psi}\right) \\
&= 2\nabla^2\Psi + \partial_i\partial_j s^{ij} \\
G_{0j} &\simeq \bar{R}_{0j} - \tfrac{1}{2}\cancel{\eta_{0j}}\bar{\bar{R}} \\
&\simeq -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\partial_k s_j{}^k + 2\partial_0\partial_j\Psi \\
G_{ij} &\simeq \bar{R}_{ij} - \tfrac{1}{2}\eta_{ij}\bar{\bar{R}} \\
&\simeq \left(-\partial_i\partial_j(\Phi-\Psi) - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k + \partial^2\Psi\delta_{ij}\right) \\
&\qquad -\tfrac{1}{2}\delta_{ij}2\left(-\nabla^2(\Phi-2\Psi) - \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_k\partial_l s^{kl} - 3{\partial_0}^2\Psi\right) \\
&= -\partial_i\partial_j(\Phi-\Psi) - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k + \delta_{ij}(-{\partial_0}^2 + \nabla^2)\Psi \\
&\qquad \delta_{ij}\nabla^2(\Phi-2\Psi) + \delta_{ij}\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \delta_{ij}\partial_k\partial_l s^{kl} + 3\delta_{ij}{\partial_0}^2\Psi \\
&= -\partial_i\partial_j(\Phi-\Psi) - \partial_0\partial_{(i}w_{j)} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k - \delta_{ij}{\partial_0}^2\Psi \\
&\qquad \delta_{ij}\nabla^2(\Phi-\Psi) + \delta_{ij}\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \delta_{ij}\partial_k\partial_l s^{kl} + 3\delta_{ij}{\partial_0}^2\Psi \\
&= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\Psi) + \delta_{ij}\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \partial_0\partial_{(i}w_{j)} - \delta_{ij}\partial_k\partial_l s^{kl} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k + 2\delta_{ij}{\partial_0}^2\Psi
\end{aligned}
$$

4.1 Dimensional Analysis

Note that the Christoffel connection[6] (symbol) Γ^i_{jk} has dimensions of inverse length and that the Riemann tensor $R^\rho{}_{\mu\sigma\nu}$, Ricci tensor $\bar{R}_{\mu\nu}$, Ricci scalar $\bar{\bar{R}}$, and Einstein tensor $G_{\mu\nu}$ all have dimensions of inverse area.

5 The Linearized Einstein Field Equation

The Einstein field equation(s) is (are)

$$G_{\mu\nu} = \frac{8\pi G_\mathrm{N}}{c^4}T_{\mu\nu}$$

or

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

[6] Is there such a thing as the abstract Christoffel connection Γ that is distinct from the Christoffel connection (symbol) Γ^i_{jk}?

where $\kappa \equiv 8\pi G_\mathrm{N}/c^4$. Since the stress tensor is defined as $T^{\mu\nu}$, the right-hand side contains a factor quadratic in the metric: $T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$.

Now, writing this (nonlinear) field equation out in terms of the metric, we get

$$\begin{aligned}
G_{\mu\nu} &= \bar{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \bar{\bar{R}} \\
&= \bar{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \bar{R}^{\mu}{}_{\mu} \\
&= R^{\rho}{}_{\mu\rho\nu} - \tfrac{1}{2} g_{\mu\nu} R^{\rho}{}_{\mu\rho\mu} \\
&= \left\{ \partial_\rho \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu} \right\} - \tfrac{1}{2} g_{\mu\nu} \left\{ \partial_\rho \Gamma^{\rho}_{\mu\mu} - \partial_\mu \Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\mu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\mu} \right\} \\
&= \tfrac{1}{2} \Big\{ \partial_\rho \left[g^{\rho\tau} (\partial_\nu g_{\mu\tau} + \partial_\mu g_{\tau\nu} - \partial_\tau g_{\nu\mu}) \right] - \partial_\nu \left[g^{\rho\tau} (\partial_\rho g_{\mu\tau} + \partial_\mu g_{\tau\rho} - \partial_\tau g_{\rho\mu}) \right] \\
&\qquad + \tfrac{1}{2} g^{\rho\tau} (\partial_\rho g_{\lambda\tau} + \partial_\lambda g_{\tau\rho} - \partial_\tau g_{\rho\lambda}) g^{\lambda\tau} (\partial_\nu g_{\mu\tau} + \partial_\mu g_{\tau\nu} - \partial_\tau g_{\nu\mu}) \\
&\qquad - \tfrac{1}{2} g^{\rho\tau} (\partial_\nu g_{\lambda\tau} + \partial_\lambda g_{\tau\nu} - \partial_\tau g_{\nu\lambda}) g^{\lambda\tau} (\partial_\rho g_{\mu\tau} + \partial_\mu g_{\tau\rho} - \partial_\tau g_{\rho\mu}) \Big\} \\
&\quad - \tfrac{1}{4} g_{\mu\nu} \Big\{ \partial_\rho \left[g^{\rho\tau} (\partial_\mu g_{\mu\tau} + \partial_\mu g_{\tau\mu} - \partial_\tau g_{\mu\mu}) \right] - \partial_\mu \left[g^{\rho\tau} (\partial_\rho g_{\mu\tau} + \partial_\mu g_{\tau\rho} - \partial_\tau g_{\rho\mu}) \right] \\
&\qquad + \tfrac{1}{2} g^{\rho\tau} (\partial_\rho g_{\lambda\tau} + \partial_\lambda g_{\tau\rho} - \partial_\tau g_{\rho\lambda}) g^{\lambda\tau} (\partial_\mu g_{\mu\tau} + \partial_\mu g_{\tau\mu} - \partial_\tau g_{\mu\mu}) \\
&\qquad - \tfrac{1}{2} g^{\rho\tau} (\partial_\mu g_{\lambda\tau} + \partial_\lambda g_{\tau\mu} - \partial_\tau g_{\mu\lambda}) g^{\lambda\tau} (\partial_\rho g_{\mu\tau} + \partial_\mu g_{\tau\rho} - \partial_\tau g_{\rho\mu}) \Big\} \\
&= \kappa g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}
\end{aligned}$$

If we take our weak-field approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\left| h_{\mu\nu} \right| \ll 1$ for some appropriately chosen coordinate system, and linearize the Einstein equation, we get

$$\begin{aligned}
G_{\mu\nu} &\simeq \tfrac{1}{2} \left\{ \eta^{\rho\tau} \partial_\rho (\partial_\nu h_{\mu\tau} + \partial_\mu h_{\tau\nu} - \partial_\tau h_{\nu\mu}) - \eta^{\rho\tau} \partial_\nu (\partial_\rho h_{\mu\tau} + \partial_\mu h_{\tau\rho} - \partial_\tau h_{\rho\mu}) \right\} \\
&= \tfrac{1}{2} \left(\partial_\rho \partial_\mu h^{\rho}{}_{\nu} + \partial_\rho \partial_\nu h^{\rho}{}_{\mu} - \partial_\mu \partial_\nu \bar{h} - \partial^2 h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \partial^2 \bar{h} \right) \\
&\simeq \kappa \left(\eta_{\mu\alpha} \eta_{\nu\beta} + \cancel{h_{\mu\alpha} \eta_{\nu\beta}} + \cancel{\eta_{\mu\alpha} h_{\nu\beta}} + \cancel{h_{\mu\alpha} h_{\nu\beta}} \right) T^{\alpha\beta} \\
&= \kappa \eta_{\mu\alpha} \eta_{\nu\beta} T^{\alpha\beta}
\end{aligned}$$

where the second line is obtained from the work done in the previous section. Again using our previous work, we can write the linearized Einstein equations in terms of the potentials:

$$\begin{aligned}
G_{00} &\simeq 2\nabla^2 \Psi + \partial_i \partial_j s^{ij} \\
&\simeq \kappa T^{00} \qquad (5) \\
G_{0j} &\simeq -\tfrac{1}{2} \nabla^2 w_j + \tfrac{1}{2} \partial_j \boldsymbol{\nabla} \cdot \mathbf{w} + \partial_0 \partial_k s_j{}^k + 2\partial_0 \partial_j \Psi \\
&\simeq -\kappa T^{0j} \qquad (6) \\
G_{ij} &\simeq (\delta_{ij} \nabla^2 - \partial_i \partial_j)(\Phi - \Psi) + \delta_{ij} \partial_0 \boldsymbol{\nabla} \cdot \mathbf{w} - \partial_0 \partial_{(i} w_{j)} - \delta_{ij} \partial_k \partial_l s^{kl} - \partial^2 s_{ij} + 2\partial_k \partial_{(i} s_{j)}{}^k + 2\delta_{ij} \partial_0{}^2 \Psi \\
&\simeq \kappa T^{ij} \qquad (7)
\end{aligned}$$

These equations are reëxpressed in particular gauges at the end of the next section, after the parallels with electromagnetism are drawn.

5.1 Dimensional Analysis

Consider

$$G_{\mu\nu} = \frac{8\pi G_\mathrm{N}}{c^4} g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta} = \kappa g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}$$

Note that since $G_{\mu\nu}$ has dimensions of inverse area, $G_{\rm N}$ has dimensions of force-area per squared mass, κ has dimensions of (force)(time)^4/(area)(mass)^2, and $g_{\mu\nu}$ is dimensionless, $T^{\mu\nu}$ must have dimensions of energy per volume (energy density), which is the same as force per area (pressure):

$$[\kappa][g_{\mu\alpha}][g_{\nu\beta}][T^{\alpha\beta}] = \left(\frac{\text{(force)(time)}^4}{\text{(area)(mass)}^2}\right)(1)(1)\left(\frac{\text{force}}{\text{area}}\right) = \left(\frac{\text{force}}{\text{(force)}^2}\right)\left(\frac{\text{force}}{\text{area}}\right) = \frac{1}{\text{(area)}} = [G_{\mu\nu}]$$

6 Gravito-Electromagnetism

If we impose two certain conditions on the matter and energy in spacetime, we will see that the Einstein and Maxwell equations are almost exactly the same. Given that we have

(1) low mass-energy density or weak gravitational fields (so we may linearize the Einstein equations)

(2) slow changes[7] in matter-energy (so we may neglect second order time derivatives)

we will find the similarity after some mathematical manipulation.

The linear Einstein equations are

$$\begin{aligned}
G_{00} &\simeq 2\nabla^2\Psi + \partial_i\partial_j s^{ij} \\
&\simeq \kappa T^{00} \\
G_{0j} &\simeq -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j \boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\partial_k s_j{}^k + 2\partial_0\partial_j\Psi \\
&\simeq -\kappa T^{0j} \\
G_{ij} &\simeq (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\Psi) + \delta_{ij}\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \partial_0\partial_{(i}w_{j)} - \delta_{ij}\partial_k\partial_l s^{kl} - \partial^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k + \cancel{2\delta_{ij}\partial_0{}^2\Psi} \\
&\simeq \kappa T^{ij}
\end{aligned}$$

If we define $\mathcal{S}^k \equiv \partial_l s^{lk}$, so that $\partial_i\partial_j s^{ij} = \partial_i\mathcal{S}^i = \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}}$, and $\partial_k s_j{}^k = \mathcal{S}_j$, and $\partial_k\partial_{(i}s_{j)}{}^k = \partial_{(i}\mathcal{S}_{j)}$, then we may rewrite the above equations as

$$\begin{aligned}
G_{00} &\simeq 2\nabla^2\Psi + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} \\
&\simeq \kappa T^{00} \\
G_{0j} &\simeq -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j \boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\mathcal{S}_j + 2\partial_0\partial_j\Psi \\
&\simeq -\kappa T^{0j} \\
G_{ij} &\simeq (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\Psi) + \delta_{ij}\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \partial_0\partial_{(i}w_{j)} - \delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} - \partial^2 s_{ij} + 2\partial_{(i}\mathcal{S}_{j)} + \cancel{2\delta_{ij}\partial_0{}^2\Psi} \\
&\simeq \kappa T^{ij}
\end{aligned}$$

Taking the trace of the third equation and defining $\langle p\rangle \equiv \frac{1}{3}T^i{}_i$, we see we have

$$\begin{aligned}
G^i{}_i &\simeq (3\nabla^2 - \nabla^2)(\Phi-\Psi) + 3\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - 3\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} - \cancel{\partial^2 s} + 2\partial^{(i}\mathcal{S}_{i)} \\
&= (3\nabla^2 - \nabla^2)(\Phi-\Psi) + 2\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - 3\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} + 2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} \\
&= 2\nabla^2(\Phi-\Psi) + 2\partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} \\
&\simeq \kappa T^i{}_i \\
&= \kappa 3\langle p\rangle
\end{aligned}$$

[7] To be more accurate, we should use a word like "acceleration" rather than change, since "slow change" implies small first derivative with respect to time.

so that $\nabla^2\Psi = \nabla^2\Phi + \partial_0 \boldsymbol{\nabla}\cdot\mathbf{w} - \frac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} - \kappa\frac{3}{2}\langle p\rangle$, and therefore $\boldsymbol{\nabla}\Psi = \boldsymbol{\nabla}\Phi + \partial_0\mathbf{w} - \frac{1}{2}\boldsymbol{\mathcal{S}} - \frac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\mathcal{P}} - \frac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\times\mathbf{C}$, where ε_g is a constant to be defined later, $\mathbf{C}$ is an arbitrary vector field, and

$$\tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\mathcal{P}}(\mathbf{r}) \equiv \frac{1}{4\pi}\int_{\mathbb{R}^3} \kappa\tfrac{3}{2}\langle p\rangle(\mathbf{r}')\frac{\mathbf{R}}{R^3}\,\mathrm{d}v' \qquad \text{so that} \qquad \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} = \kappa\tfrac{3}{2}\langle p\rangle$$

with $R = |\mathbf{R}| \equiv \left|\mathbf{r}-\mathbf{r}'\right|$. Then, rewriting the equations for G_{00} and G_{0j},

$$\begin{aligned} 2\left(\nabla^2\Phi + \partial_0\boldsymbol{\nabla}\cdot\mathbf{w} - \cancel{\tfrac{1}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}}} - \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}}\right) + \cancel{\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}}} &\simeq \kappa T^{00} \\ -\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} + \partial_0\mathcal{S}_j + 2\partial_0\left(\partial_j\Phi + \partial_0 w_j - \tfrac{1}{2}\mathcal{S}_j - \tfrac{1}{\varepsilon_\mathrm{g}}\mathcal{P}_j - \tfrac{1}{\varepsilon_\mathrm{g}}(\boldsymbol{\nabla}\times\mathbf{C})_j\right) &\simeq -\kappa T^{0j} \end{aligned}$$

$$-\tfrac{1}{2}\nabla^2 w_j + \tfrac{1}{2}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} + \cancel{\partial_0\mathcal{S}_j} + 2\partial_0\partial_j\Phi + \cancel{2\partial_0{}^2 w_j} - \cancel{\partial_0\mathcal{S}_j} \simeq -\kappa T^{0j} + 2\tfrac{1}{\varepsilon_\mathrm{g}}\partial_0(\boldsymbol{\nabla}\times\mathbf{C})_j + 2\tfrac{1}{\varepsilon_\mathrm{g}}\partial_0\mathcal{P}_j$$

or, rearranging and dividing by 2,

$$\begin{aligned} -\nabla^2\Phi - \tfrac{1}{c}\partial_t\boldsymbol{\nabla}\cdot\mathbf{w} &\simeq -\tfrac{4\pi G_\mathrm{N}}{c^4}T^{00} - \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} \\ \tfrac{1}{c}\partial_t\partial_j\Phi + \tfrac{1}{4}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} - \tfrac{1}{4}\nabla^2 w_j &\simeq -\tfrac{4\pi G_\mathrm{N}}{c^4}T^{0j} + \tfrac{1}{\varepsilon_\mathrm{g}}\tfrac{1}{c}\partial_t(\boldsymbol{\nabla}\times\mathbf{C})_j + \tfrac{1}{\varepsilon_\mathrm{g}}\tfrac{1}{c}\partial_t\mathcal{P}_j \end{aligned}$$

Now let's take a look at the electromagnetic potential equations and see how similar they look to the two gravitational potential equations immediately above.

$$\begin{aligned} -\nabla^2 V - \partial_t\boldsymbol{\nabla}\cdot\mathbf{A} &= \tfrac{1}{\varepsilon_0}\rho - \tfrac{1}{\varepsilon_0}\boldsymbol{\nabla}\cdot\mathbf{P} \\ \tfrac{1}{c^2}\partial_t\boldsymbol{\nabla}V + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \partial^2\mathbf{A} &= \mu_0\mathbf{J} + \mu_0\boldsymbol{\nabla}\times\mathbf{M} + \mu_0\partial_t\mathbf{P} \\ \tfrac{1}{c^2}\partial_t\partial_j V + \partial_j\boldsymbol{\nabla}\cdot\mathbf{A} - \partial^2 A_j &= \mu_0 J_j + \mu_0(\boldsymbol{\nabla}\times\mathbf{M})_j + \mu_0\partial_t P_j \end{aligned}$$

To make the gravitational equations more like the electromagnetic ones, we can add $\frac{1}{4}\partial_0{}^2 w_j$ on the left of the second equation (which will combine with the $-\nabla^2 w_j$ term to yield a $-\partial^2 w_j$ term) since it is negligible by assumption, and let's define

$$\boldsymbol{\mathcal{M}} \equiv \partial_t\mathbf{C}$$

which is just as arbitrary as the arbitrary vector field $\mathbf{C}$.

And now, let's finish the transformation by figuring out how the constants should relate. Using Newton's law of gravitation and Coulomb's law, we compare $F = K_\mathrm{e}q_1q_2/r^2$ and $F = G_\mathrm{N}m_1m_2/r^2$ and see $G_\mathrm{N} = K_\mathrm{ge} = 1/4\pi\varepsilon_\mathrm{g}$. Since $\varepsilon_0\mu_0 = c^{-2} = K_\mathrm{m}/K_\mathrm{e}$, we have $G_\mathrm{N} = c^2 K_\mathrm{gm} = c^2\mu_\mathrm{g}/4\pi$. So the gravitation equations become (after dividing the second one by c)

$$\begin{aligned} -\nabla^2\Phi - \tfrac{1}{c}\partial_t\boldsymbol{\nabla}\cdot\mathbf{w} &\simeq -\tfrac{1}{c^4\varepsilon_\mathrm{g}}T^{00} - \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} \\ \tfrac{1}{c^2}\partial_t\partial_j\Phi + \tfrac{1}{4c}\partial_j\boldsymbol{\nabla}\cdot\mathbf{w} - \tfrac{1}{4c}\partial^2 w_j &\simeq -\tfrac{c^2\mu_\mathrm{g}}{c^5}T^{0j} + \tfrac{1}{\varepsilon_\mathrm{g}}\tfrac{1}{c^2}(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}})_j + \tfrac{1}{\varepsilon_\mathrm{g}}\tfrac{1}{c^2}\partial_t\mathcal{P}_j \end{aligned}$$

A final rearranging yeilds a Maxwellian form of the (slowly changing) linear Einstein potential field equations

$$-\nabla^2\Phi - \partial_t\boldsymbol{\nabla}\cdot\left(\tfrac{\mathbf{w}}{c}\right) \simeq -\tfrac{1}{\varepsilon_\mathrm{g}}\left(\tfrac{T^{00}}{c^4}\right) - \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} \tag{8}$$

$$\tfrac{1}{c^2}\partial_t\partial_j\Phi + \tfrac{1}{4}\left[\partial_j\boldsymbol{\nabla}\cdot\left(\tfrac{\mathbf{w}}{c}\right) - \partial^2\left(\tfrac{w_j}{c}\right)\right] \simeq -\mu_\mathrm{g}\left(c\tfrac{T^{0j}}{c^4}\right) + \mu_\mathrm{g}(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}})_j + \mu_\mathrm{g}\partial_t\mathcal{P}_j \tag{9}$$

which you can see are very similar to the Maxwell potential field equations

$$-\nabla^2 V - \partial_t\boldsymbol{\nabla}\cdot\mathbf{A} = \tfrac{1}{\varepsilon_0}\rho^\mathrm{f} - \tfrac{1}{\varepsilon_0}\boldsymbol{\nabla}\cdot\mathbf{P} \tag{10}$$

$$\tfrac{1}{c^2}\partial_t\partial_j V + \partial_j\boldsymbol{\nabla}\cdot\mathbf{A} - \partial^2 A_j = \mu_0 J_j^\mathrm{f} + \mu_0(\boldsymbol{\nabla}\times\mathbf{M})_j + \mu_0\partial_t P_j \tag{11}$$

Of course, if we define the gravito-electric field $\mathbf{G}$ and the gravito-magnetic field $\mathbf{H}$

$$\mathbf{G} \equiv -\boldsymbol{\nabla}\Phi - \tfrac{1}{c}\partial_t \mathbf{w} \qquad \text{and} \qquad \mathbf{H} \equiv \boldsymbol{\nabla}\times\mathbf{w}$$

as analogues to the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$

$$\mathbf{E} = -\boldsymbol{\nabla}V - \partial_t \mathbf{A} \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

and we define the "free" (non-gravitational) energy[8] density $\rho^{(\mathrm{e})\mathrm{f}}$ and "free" energy current density $\mathbf{J}^{(\mathrm{e})\mathrm{f}}$ by

$$\rho^{(\mathrm{e})\mathrm{f}} \equiv \frac{T^{00}}{c^4} \qquad \text{and} \qquad (\mathbf{J}^{(\mathrm{e})\mathrm{f}})^j \equiv c\frac{T^{0j}}{c^4}$$

then we gain the gravito-electromagnetic field equations

$$\boldsymbol{\nabla}\cdot\mathbf{G} \simeq -\tfrac{1}{\varepsilon_\mathrm{g}}\rho^{(\mathrm{e})\mathrm{f}} - \tfrac{1}{\varepsilon_\mathrm{g}}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}} \tag{12}$$

$$\tfrac{1}{4c}\boldsymbol{\nabla}\times\mathbf{H} - \tfrac{1}{c^2}\partial_t\mathbf{G} \simeq -\mu_\mathrm{g}\mathbf{J}^{(\mathrm{e})\mathrm{f}} + \mu_\mathrm{g}\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}} + \mu_\mathrm{g}\partial_t\boldsymbol{\mathcal{P}} \tag{13}$$

very much like the electromagnetic field equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \tfrac{1}{\varepsilon_0}\rho^\mathrm{f} - \tfrac{1}{\varepsilon_0}\boldsymbol{\nabla}\cdot\mathbf{P} \tag{14}$$

$$\boldsymbol{\nabla}\times\mathbf{B} - \tfrac{1}{c^2}\partial_t\mathbf{E} = \mu_0\mathbf{J}^\mathrm{f} + \mu_0\boldsymbol{\nabla}\times\mathbf{M} + \mu_0\partial_t\mathbf{P} \tag{15}$$

The factor of $1/4c$ arises since

$$\begin{aligned}
\tfrac{1}{4c}\boldsymbol{\nabla}\times\mathbf{H} - \tfrac{1}{c^2}\partial_t\mathbf{G} &\simeq \tfrac{1}{4c}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{w} - \tfrac{1}{c^2}\partial_t\left(-\boldsymbol{\nabla}\Phi - \tfrac{1}{c}\partial_t\mathbf{w}\right) \\
&= \tfrac{1}{4c}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{w}) - \tfrac{1}{4c}\boldsymbol{\nabla}^2\mathbf{w} + \tfrac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi + \cancel{\tfrac{1}{c^3}\partial_t^2\mathbf{w}} \\
&\simeq \tfrac{1}{4}\left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\tfrac{\mathbf{w}}{c}) - \boldsymbol{\nabla}^2\tfrac{\mathbf{w}}{c}\right] + \tfrac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi \\
&\simeq \tfrac{1}{4}\left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\tfrac{\mathbf{w}}{c}) - \partial^2\tfrac{\mathbf{w}}{c}\right] + \tfrac{1}{c^2}\partial_t\boldsymbol{\nabla}\Phi \\
&\simeq -\mu_\mathrm{g}\mathbf{J}^{(\mathrm{e})} + \mu_\mathrm{g}\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}} + \mu_\mathrm{g}\partial_t\boldsymbol{\mathcal{P}}
\end{aligned}$$

So, the equations are very similar, but what do the differences mean? One difference is the changes in signs, which is easily explained: in electromagnetism like charges repel but in gravitation like charges (positive masses) attract. Another difference is presence of the factors of c. I'm not yet sure how to explain this. [9] There is also the factor of 4: what Wikipedia says in the article on Gravitoelectromagnetism[10] seems to suggest that this has to do with the spin of the graviton. Finally, there is the task of discovering in what sense $\rho^{(\mathrm{e})\mathrm{f}}$ and $\mathbf{J}^{(\mathrm{e})\mathrm{f}}$ are "free", interpreting what $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{M}}$ mean, and determining why $\boldsymbol{\mathcal{M}}$ is arbitrary (or at least seems to be arbitrary). Apparently, $\boldsymbol{\mathcal{P}}$ has to do with "bound" energy, but I'm not sure how to check that yet. And $\boldsymbol{\mathcal{M}}$ appears to do with "bound" circulating mass, or angular momentum, perhaps including quantum mechanical inertial spin, if it exists.

6.1 Differences Between EM and GEM Equations

(1) signs (gravitation is only attractive, mass is only positive)

(2) factors of c (trivial?)

(3) factor of 4 (spin-2 graviton?)

[8] Since T^{00} and T^{0j} have dimensions of energy density, this terminology of "energy density" is not quite right.

[9] We may have to do some constant shuffling to get a more ideal and analogous combination. It may be that the SI definition of $\mathbf{B}$ or $\mathbf{A}$ is not ideal since we have $\mathbf{E} = -\boldsymbol{\nabla}V - \partial_t\mathbf{A}$ rather than $\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\partial_t\mathbf{A}$.

[10] Wikipedia "Gravitoelectromagnetism" article URI: `http://en.wikipedia.org/wiki/Gravitoelectromagnetism`

6.2 Issues to Resolve

(1) the meaning of $\rho^{(\mathrm{e})\mathrm{f}}$ and $\mathbf{J}^{(\mathrm{e})\mathrm{f}}$ ("free"?)

(2) the meaning of $\boldsymbol{\mathcal{P}}$ ("bound" energy?)

(3) the meaning of the seemingly arbitrary $\boldsymbol{\mathcal{M}}$ ("bound" angular momentum, spin?)

6.3 Gauge Freedom in Potentials

Let's compare the gravitational and electromagnetic gauge transformations (using the gravitational gauge[11] $\xi^\mu(t,x)$ and the electromagnetic gauge $\lambda(t,x)$) and then look at some particular gauge choices.

$$\begin{aligned} \Phi &\to \Phi + \partial_0 \xi^0 & \qquad V &\to V - \partial_t \lambda \\ w_i &\to w_i + \partial_0 \xi^i - \partial_i \xi^0 & \qquad A^i &\to A^i + \partial^i \lambda \\ \Psi &\to \Psi - \tfrac{1}{3}\partial_i \xi^i & & \\ s_{ij} &\to s_{ij} + \partial_{(i}\xi_{j)} - \tfrac{1}{3}\partial_k \xi^k \delta_{ij} & & \end{aligned}$$

- Transverse Gauge: (generalization of the conformal Newtonian or Poisson gauge, closely related to Coulomb gauge in electrodynamics)

$$\begin{aligned} \partial_i w^i &= 0 \qquad \text{(analogue of Coulomb gauge, } \partial_i A^i = 0\text{)} \\ \partial_i s^{ij} &= 0 \end{aligned}$$

  Linearized Einstein Field Equations in Transverse Gauge:

$$\begin{aligned} G_{00} &\simeq 2\nabla^2 \Psi \\ &\simeq \kappa T^{00} \\ G_{0j} &\simeq -\tfrac{1}{2}\nabla^2 w_j + 2\partial_0 \partial_j \Psi \\ &\simeq -\kappa T^{0j} \\ G_{ij} &\simeq (\delta_{ij}\nabla^2 - \partial_i \partial_j)(\Phi - \Psi) - \partial_0 \partial_{(i} w_{j)} + 2\delta_{ij}{\partial_0}^2 \Psi - \partial^2 s_{ij} \\ &\simeq \kappa T^{ij} \end{aligned}$$

- Synchronous gauge: (equivalent to Gaussian normal coordinates, "as discussed in Appendix D"; gravitational analogue of the temporal gauge of electrodynamics $A^0 = 0$, since it kills off the nonspatial components of the perturbation)

$$\Phi = 0$$

- Lorenz / harmonic gauge: (note $\bar{h} \simeq \eta^{\mu\nu} h_{\mu\nu}$)

$$\partial_\mu h^\mu{}_\nu - \tfrac{1}{2}\partial_\nu \bar{h} = 0$$

7 The Geodesic Equation and Gravito-Lorentz Force

A test particle in free-fall (where there are only gravitational forces) will follow a geodesic:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0$$

[11] Is ξ^μ a "4-gauge"?

We also have that

$$\begin{aligned}
p^\mu &= \tfrac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = m\tfrac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \\
p^0 &= c\tfrac{\mathrm{d}t}{\mathrm{d}\lambda} = E/c \\
p^i &= m\tfrac{\mathrm{d}t}{\mathrm{d}\tau}\tfrac{\mathrm{d}x^i}{\mathrm{d}t} = \gamma m v^i = Ev^i/c^2
\end{aligned}$$

where λ is an affine parameter (related to the proper time by an affine transformation), and $\lambda = \tau/m$ and $E = \gamma mc^2$ if the particle is massive. [12]

So the geodesic equation can be rewritten, emulating Newton's second law,

$$\begin{aligned}
\tfrac{\mathrm{d}p^\mu}{\mathrm{d}t} &= \tfrac{\mathrm{d}\lambda}{\mathrm{d}t}\tfrac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} \\
&= -\frac{c^2}{E}\Gamma^\mu_{\rho\sigma}p^\rho p^\sigma \\
\tfrac{\mathrm{d}E}{\mathrm{d}t} &= c\tfrac{\mathrm{d}p^0}{\mathrm{d}t} = -\frac{c^3}{E}\Gamma^0_{\rho\sigma}p^\rho p^\sigma \\
&= -\frac{c^3}{E}\left[\Gamma^0_{00}p^0p^0 + 2\Gamma^0_{j0}p^jp^0 + \Gamma^0_{jk}p^jp^k\right] \\
&= -\frac{c^3}{E}\left[(\partial_0\Phi)(E/c)^2 + 2(\partial_j\Phi)(Ev^j/c^2)(E/c) + \left(-\partial_{(j}w_{k)} + \tfrac{1}{2}\partial_0 h_{jk}\right)(Ev^j/c^2)(Ev^k/c^2)\right] \\
&= -cE\left[\partial_0\Phi + 2(\partial_j\Phi)v^j/c - \left(\partial_{(j}w_{k)} - \tfrac{1}{2}\partial_0 h_{jk}\right)v^jv^k/c^2\right] \\
\tfrac{\mathrm{d}p^i}{\mathrm{d}t} &= -\frac{c^2}{E}\Gamma^i_{\rho\sigma}p^\rho p^\sigma \\
&= -\frac{c^2}{E}\left[\Gamma^i_{00}p^0p^0 + 2\Gamma^i_{j0}p^jp^0 + \Gamma^i_{jk}p^jp^k\right] \\
&= -\frac{c^2}{E}\Big[(\partial_i\Phi + \partial_0 w_i)(E/c)^2 + 2\left(\partial_{[j}w_{i]} + \tfrac{1}{2}\partial_0 h_{ij}\right)(Ev^j/c^2)(E/c) \\
&\qquad + \left(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right)(Ev^j/c^2)(Ev^k/c^2)\Big] \\
&= -E\left[\partial_i\Phi + \partial_0 w_i + 2\left(\partial_{[j}w_{i]} + \tfrac{1}{2}\partial_0 h_{ij}\right)v^j/c + \left(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right)v^jv^k/c^2\right]
\end{aligned}$$

This expression for $\mathrm{d}p^i/\mathrm{d}t$ simply shows the different kinds of forces that an observer will be able to detect in his coordinate frame. Now, we may use gravito-electromagnetic field definitions to reëxpress these forces.

$$\mathbf{G} \equiv -\boldsymbol{\nabla}\Phi - \tfrac{1}{c}\partial_t\mathbf{w} \qquad G^i \equiv -\partial^i\Phi - \partial_0 w^i$$
$$\mathbf{H} \equiv \boldsymbol{\nabla}\times\mathbf{w} \qquad H^i \equiv \tilde{\varepsilon}^{ijk}\partial_j w_k$$

and since

$$\begin{aligned}
H^i &= \tilde{\varepsilon}^{iab}(\delta^{jk}_{ab} - \delta^{jk}_{ba})\partial_j w_k = \tilde{\varepsilon}^{iab}(\partial_a w_b - \partial_b w_a) \\
(\mathbf{v}\times\mathbf{H})^i &= \tilde{\varepsilon}^{ijk}v_jH_k \overset{\text{abuse}}{=} \tilde{\varepsilon}^{ijk}v_jH^k \\
&= \tilde{\varepsilon}^{ijk}v_j\tilde{\varepsilon}^{klm}\partial_l w_m = \tilde{\varepsilon}^{klm}\tilde{\varepsilon}^{kij}\partial_l w_m v_j = (\delta^{lm}_{ij} - \delta^{lm}_{ji})\partial_l w_m v_j \\
&= (\partial_i w_j - \partial_j w_i)v_j \overset{\text{abuse}}{=} (\partial_i w_j - \partial_j w_i)v^j \\
&= (\mathbf{v}\times\boldsymbol{\nabla}\times\mathbf{w})^i \\
\partial_{[j}w_{i]}v^j &= \tfrac{1}{2}(\partial_j w_i - \partial_i w_j)v^j = -\tfrac{1}{2}(\mathbf{v}\times\mathbf{H})^i
\end{aligned}$$

[12] What about light-like trajectories? What kind of parametrizations are possible?

we can restate and reformulate the geodesic equations like so:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -E\left[\partial_0\Phi + 2(\partial_j\Phi)v^j/c - \left(\partial_{(j}w_{k)} - \tfrac{1}{2}\partial_0 h_{jk}\right)v^j v^k/c^2\right] \tag{16}$$

$$\begin{aligned}\frac{\mathrm{d}p^i}{\mathrm{d}t} &= E\left[-\partial_i\Phi - \partial_0 w_i - 2\partial_{[j}w_{i]}v^j/c - \partial_0 h_{ij}v^j/c - \left(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right)v^j v^k/c^2\right] \\ &= E\left[G^i + \left(\tfrac{\mathbf{v}}{c}\times\mathbf{H}\right)^i - (\partial_0 h_{ij})v^j/c - \left(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right)v^j v^k/c^2\right]\end{aligned} \tag{17}$$

Notice that for very slowly changing (in space and time) h_{ij}, Equation (17) is nearly identical to the electromagnetic Lorentz force law:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{gL}} = E\left[\mathbf{G} + \left(\tfrac{\mathbf{v}}{c}\times\mathbf{H}\right)\right]$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{L}} = Q\left[\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right]$$

The only difference this time is the factor of c in the magnetic term of the gravito-Lorentz force.

7.1 Dimensional Analysis

Note that we obtain the gravito-electric force

$$\mathbf{F}_{\mathrm{ge}} = E\mathbf{G}$$

(where $E = \gamma mc^2$ as defined earlier) from the gravito-Lorentz force and that this agrees dimensionally with our definition of $\mathbf{G}$

$$\mathbf{G} \equiv -\boldsymbol{\nabla}\Phi - \tfrac{1}{c}\partial_t\mathbf{w}$$

which shows that $\mathbf{G}$ has dimensions of inverse length since Φ and $\mathbf{w}$ are dimensionless. [13]

[13] If $\partial_t\mathbf{w} = \mathbf{0}$, the gravitational potential Φ tells you how much gravitational energy there is per (non-gravitational) energy in spacetime. Correct? (How does the feedback or nonlinear idea of gravitational-field-becomes-source come into play?)

8 A Simple Geometry: The Lense-Thirring Effect

Exercise 7.2 in Carroll [1] (pg 320): Consider a thin spherical shell of matter, with mass M and radius R, slowly rotating with an angular velocity Ω.

(a) Show $\mathbf{G} = \mathbf{0}$ (inside the shell) and calculate $\mathbf{H} = \mathbf{H}(M, R, \Omega)$.

(b) $\mathbf{H} \neq \mathbf{0} \Rightarrow$ inertial frame dragging (Lense-Thirring effect)
Calculate the rotation (relative to the background Minkowski inertial frame) of a freely-falling observer sitting at the center of the shell.

Solution to Part (a)

- Slowly rotating $\Rightarrow R\Omega \ll c$
- We assume low mass-energy density
  $\Rightarrow$ GEM eqns valid
  $\Rightarrow$ We can use dust stress tensor (eqn 1.110): $T^{\mu\nu} = \rho u^\mu u^\nu$
- From the shell, we have $\rho = \rho_0\, \delta(r - R) = \frac{Mc^2}{4\pi R^2}\, \delta(r - R)$

$$\begin{aligned}
u^\mu &= \gamma_v (c, \mathbf{v})^\mu \qquad \text{where } v = R\Omega\, \mathrm{S}\theta \text{ so } \gamma_v \simeq 1 \\
&\simeq \big(c, R\Omega\, \mathrm{S}\theta\, \langle -\mathrm{S}\phi, \mathrm{C}\phi, 0\rangle\big)^\mu \\
&= c\left(1, \tfrac{R\Omega}{c}\, \mathrm{S}\theta\, \langle -\mathrm{S}\phi, \mathrm{C}\phi, 0\rangle\right)^\mu
\end{aligned}$$

$$\begin{aligned}
T^{\mu\nu} &= \frac{Mc^4}{4\pi R^2}\, \delta(r - R) \begin{pmatrix}
1 & -\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{S}\phi & \frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{C}\phi & 0 \\
-\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{S}\phi & (\frac{R\Omega}{c})^2\, \mathrm{S}^2\theta\, \mathrm{S}^2\phi & -(\frac{R\Omega}{c})^2\, \mathrm{S}^2\theta\, \mathrm{C}\phi\, \mathrm{S}\phi & 0 \\
\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{C}\phi & -(\frac{R\Omega}{c})^2\, \mathrm{S}^2\theta\, \mathrm{C}\phi\, \mathrm{S}\phi & (\frac{R\Omega}{c})^2\, \mathrm{S}^2\theta\, \mathrm{C}^2\phi & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
&= \frac{Mc^4}{4\pi R^2}\, \delta(r - R) \begin{pmatrix}
1 & -\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{S}\phi & \frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{C}\phi & 0 \\
-\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{S}\phi & 0 & 0 & 0 \\
\frac{R\Omega}{c}\, \mathrm{S}\theta\, \mathrm{C}\phi & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}$$

Consider the linearized Einstein field equations in the transverse gauge, Equations (7.38)–(7.40):

$$\begin{aligned}
G_{00} &= 2\nabla^2\Psi = \kappa T^{00} \\
&= 8\pi G_\mathrm{N} \frac{M}{4\pi R^2}\, \delta(r - R) \\
G_{0j} &= -\tfrac{1}{2}\nabla^2 w_j + \cancel{2\partial_0\partial_j\Psi} \\
&= -\kappa T^{0j} \\
G_{ij} &= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \cancel{\partial_0\partial_{(i}w_{j)}} + \cancel{2\delta_{ij}\partial_0{}^2\Psi} - \partial^2 s_{ij} \\
&= \kappa T^{ij} \\
&= 0
\end{aligned}$$

where $\kappa \equiv 8\pi G_{\rm N}/c^4$. The stress tensor $T^{\mu\nu}$ is independent of time, so everything else should be likewise and time-differentiated terms drop out.

After using separation of variables and the appropriate boundary conditions, we get the solutions for the potentials:

$$
\begin{aligned}
\Phi = \Psi &= \begin{cases} -\frac{GM}{R}, & \text{for } r < R \\ -\frac{GM}{r}, & \text{for } r \geq R \end{cases} \\
w_1 &= \begin{cases} \frac{4GM\Omega}{3R} y, & \text{for } r < R \\ \frac{4GM\Omega}{3R}\frac{R^2}{r^3} y, & \text{for } r \geq R \end{cases} \\
w_2 &= \begin{cases} -\frac{4GM\Omega}{3R} x, & \text{for } r < R \\ -\frac{4GM\Omega}{3R}\frac{R^2}{r^3} x, & \text{for } r \geq R \end{cases} \\
w_3 &= 0 \\
s_{ij} &= 0
\end{aligned}
$$

$$
\mathbf{G} = -\boldsymbol{\nabla}\Phi - \cancel{\partial_0 \mathbf{w}} = \begin{cases} \mathbf{0}, & \text{for } r < R \\ -\frac{GM}{r^3}\mathbf{r}, & \text{for } r \geq R \end{cases}
$$

Inside the shell,

$$
\begin{aligned}
\mathbf{H} &= \boldsymbol{\nabla}\times\mathbf{w} = \left\langle -\cancel{\partial_z w_2}, \cancel{\partial_z w_1}, (\partial_x w_2 - \partial_y w_1)\right\rangle \\
&= \left(-\frac{4GM\Omega}{3R} - \frac{4GM\Omega}{3R}\right)\hat{\mathbf{z}} \\
&= -\frac{8GM\Omega}{3R}\hat{\mathbf{z}}
\end{aligned}
$$

Outside the shell,

$$
\begin{aligned}
\mathbf{H} &= \boldsymbol{\nabla}\times\mathbf{w} = \left\langle -\partial_z w_2, \partial_z w_1, (\partial_x w_2 - \partial_y w_1)\right\rangle \\
&= \left\langle -\frac{4GM\Omega}{R}\frac{R^2}{r^5} zx, \frac{4GM\Omega}{R}\frac{R^2}{r^5} zy, \left(-\frac{4GM\Omega}{3R} - \frac{4GM\Omega}{3R}\right)\frac{R^2}{r^3} + \left(\frac{4GM\Omega}{R}\frac{R^2}{r^5}x^2 + \frac{4GM\Omega}{R}\frac{R^2}{r^5}y^2\right)\right\rangle \\
&= \frac{4GM\Omega}{R}\frac{R^2}{r^3}\left\langle -\frac{zx}{r^2}, \frac{zy}{r^2}, -\frac{2}{3} + \frac{(x^2+y^2)}{r^2}\right\rangle
\end{aligned}
$$

Solution to Part (b)

Inside and at the center of the shell, we have an observer of mass m with energy $E = \gamma_v mc^2$ and momentum $\mathbf{p} = m\mathbf{v}$.

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= -E\left[\cancel{\partial_0\Phi} + 2\cancel{(\partial_j\Phi)}v^j/c - \left(\partial_{(j}w_{k)} - \tfrac{1}{2}\partial_0 h_{jk}\right)\cancel{v^j v^k/c^2}\right] \\
&\simeq 0 \\
\frac{\mathrm{d}p^i}{\mathrm{d}t} &= E\left[\cancel{G^i} + \left(\frac{\mathbf{v}}{c}\times\mathbf{H}\right)^i - (\partial_0 \cancel{h_{ij}})v^j/c - \left(\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right)\cancel{v^j v^k/c^2}\right]
\end{aligned}
$$

So the energy E of the particle is essentially constant and we have

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = E\frac{\mathbf{p}}{mc} \times \mathbf{H} = \frac{E}{mc}\mathbf{p} \times \mathbf{H}$$

Since $\mathbf{H}$ is a constant vector in the $-z$-direction, we see that $\mathbf{p}$ is precessing: let $\mathbf{p}_\perp$ be the projection of $\mathbf{p}$ on the x-y plane (perpendicular to $\mathbf{H}$) so

$$\begin{aligned}
\tfrac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} &= \frac{E}{mc}\mathbf{p}_\perp \times \mathbf{H} = \mathbf{F}_{\text{centripetal}} = m r_{\text{rot}}\omega^2 \\
\tfrac{\mathrm{d}^2\mathbf{p}_\perp}{\mathrm{d}t^2} &= \frac{E}{mc}\tfrac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} \times \mathbf{H} = \left(\frac{E}{mc}\right)^2 (\mathbf{p}_\perp \times \mathbf{H}) \times \mathbf{H} \\
&= -\left(\frac{EH}{mc}\right)^2 \mathbf{p}_\perp
\end{aligned}$$

Thus, the angular frequency of the circular motion of the observer is

$$\begin{aligned}
\omega &= \frac{E}{mc}|H| = \frac{E}{mc}\frac{8GM\Omega}{3R} \\
&\simeq \frac{mc^2 + \mathbf{p}^2/2m}{mc}\frac{8GM\Omega}{3R} = c\left(1 + \mathbf{v}^2/c^2\right)\frac{8GM\Omega}{3R}
\end{aligned} \tag{18}$$

and since

$$v_\perp = m r_{\text{rot}}\omega$$

we have

$$\begin{aligned}
r_{\text{rot}} &= \frac{v_\perp}{m}\frac{1}{\omega} = \frac{v_\perp}{m}\frac{mc}{E}\frac{3R}{8GM\Omega} \\
&= \frac{v_\perp}{mc(1 + \mathbf{v}^2/c^2)}\frac{3R}{8GM\Omega}
\end{aligned} \tag{19}$$

In obtaining r_{rot} and ω we have found the rotation (relative to the background Minkowski inertial frame) of a freely-falling observer sitting at the center of the shell. Note that the rotation is dependent upon the initial velocity of the observer and there is no rotation if the observer is initially stationary.

9 Further Topics and Resources

Papers

- Brill and Cohen: *Rotating Masses and Their Effect on Inertial Frames*, Phys. Rev. 143, 1011 - 1015 (1966) [Issue 4 – March 1966]
- Cohen: *Gravitational Collapse of Rotating Bodies*, Phys. Rev. 173, 1258 - 1263 (1968) [Issue 5 – September 1968]

References

[1] Sean M. Carroll: *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley (2004)

[2] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler: *Gravitation*, W. H. Freeman and Company (1973)