Free Quantum Field Theory of Scalar Particles
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Chris Clark   Andrew Forrester

In this paper we will motivate and derive the quantum field theory of spinless particles in the absence of interactions, starting from the assumptions of quantum mechanics. As we take a more physical perspective of field theory than is usual and reason from quantum mechanical ideas rather than in analogy with them, we come closer to "proving" the existence of anti-particles, vacuum fluctuations, and the phenomenon of indistinguishability.

1 Introduction

1.1 Motivation

Why do we need quantum field theory? Quantum mechanics works just fine when we are modeling a fixed number of particles, but it does not properly describe situations in which the number of particles changes due to interactions.[1] Take annihilation for example; it might at first seem that quantum mechanics should suffice as long as we agree to set the particle wave functions to zero as soon as annihilation takes place. However, this has a crucial flaw. Since there is uncertainty with respect to the positions of the particles, there is also uncertainty as to whether the two particles have annihilated at any given time. It is not just that we are unable to measure whether annihilation has occurred; there is actually no discrete answer to the question. The reality is that the universe is in a superposition of states – in some annihilation has occurred and in others it has not. Just as quantum mechanics has a wave function to store information about the position of a particle, quantum field theory creates an object to store information about the probabilities of each possible configuration of particle wave functions: the Schrödinger functional. This strategy is known as second quantization because a probabilistic interpretation is being given to the wave functions for observables rather than the observables themselves.

In order to define the Schrödinger functional, we must first define the particle field $\phi(\mathbf{x}, t)$. The particle field encodes all the information about the particles in the universe that belong to a specific indistinguishability class. An indistinguishability class is simply a set of particles that are indistinguishable by their intrinsic properties such as mass and charge. For example, all photons are in one indistinguishability class and all electrons are in another. The method of encoding the wave functions in $\phi(\mathbf{x}, t)$ is simple; $\phi(\mathbf{x}, t)$ is just a combined wave function equal to the sum of the wave functions of all the particles (including vacuum fluctuations) in the indistinguishability class it refers to. We suggest that $\phi(\mathbf{x}, t)$ is the only physically real wave function whereas the wave functions from quantum mechanics were just a natural simplification for describing non-overlapping particles. Of course, the deficiencies of quantum mechanical wave functions can be corrected by (anti-)symmetrization, but this is simply patchwork that results from using the wrong fundamental wave function. Given this interpretation of $\phi(\mathbf{x}, t)$, which we take to be an axiom of the theory, the Schrödinger functional $\Psi[\phi]$ can be defined as the probability amplitude that the universe would be measured to be in the state characterized by $\phi(\mathbf{x}, t)$, assuming such a measurement could be made. It is not practically possible to simultaneously take a position measurement of every electron in the universe, but quantum mechanics does not forbid the possibility.

[1] Quantum mechanics can be used to model the creation and destruction of particles, but not properly; it neglects the crucial flaw discussed below. See Peskin and Schroeder page 32 for an example of how to model creation and destruction in quantum mechanics.

1.2 Formulations of Quantum Field Theory

Before continuing with the development along these lines, we would like to discuss the other prevalent formulations of quantum field theory for those readers who already have some familiarity with the subject. There are three main formulations of quantum field theory: the canonical formulation, the Feynman formulation, and the Schrödinger formulation.[2] In this paper we will be developing the Schrödinger formulation, but this is by far the least popular formulation in the literature. It is perhaps unfortunate because we believe that it is the only choice that provides a clear picture of the underlaying concepts. Studying the canonical or Feynman formulations without understanding the Schrödinger formulation is precisely analogous to studying Lagrangian dynamics without understanding Newton's laws. In both cases, some clever mathematical manipulations serve to make computations much simpler, but at the same time obscure the underlying physics. But why do we even have three different formulations? Each formulation accomplishes the same thing using different mathematical tools. What they all accomplish is that they ensure that the expectation value of the particle field ϕ obeys the proper field equation.

In this paper we will be assuming that the field equation is the Klein-Gordon equation. This is standard introductory example found in the textbooks since it describes the simplest case of spinless particles. First we will derive the Klein-Gordon equation from the Schrödinger equation.

$$\hat{H}\left|\phi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\phi(t)\right\rangle$$

Relativity states that the Hamiltonian for a free particle is $\hat{H} = \sqrt{\hat{\mathbf{p}}^2c^2 + m^2c^4}$. If we insert this Hamiltonian into Schrödinger's equation, we get

$$\sqrt{\hat{\mathbf{p}}^2c^2 + m^2c^4}\left|\phi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\phi(t)\right\rangle$$

There is no clean way to solve an equation with a differential operator inside a square root, but it is possible to fix the problem by squaring the equation, or more rigorously, we apply $\hat{H}$ to both sides and use the fact that $\frac{\partial}{\partial t}$ commutes with $\hat{H}$ because energy is conserved

$$(\hat{\mathbf{p}}^2c^2 + m^2c^4)\left|\phi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\hat{H}\left|\phi(t)\right\rangle = -\hbar^2\frac{\partial^2}{\partial t^2}\left|\phi(t)\right\rangle$$

We now convert to the coordinate representation by applying $\langle\mathbf{x}|$ to the left. Then using $\langle\mathbf{x}|\,\hat{\mathbf{p}}\left|\phi(t)\right\rangle = -i\hbar\nabla\left\langle\mathbf{x}|\,\phi(t)\right\rangle$ and $\left\langle\mathbf{x}|\,\phi(t)\right\rangle = \phi(\mathbf{x}, t)$,

$$\left(\hbar^2\frac{\partial^2}{\partial t^2} - \hbar^2c^2\nabla^2 + m^2c^4\right)\phi(\mathbf{x}, t) = 0$$

And dividing by $\hbar^2c^2$ gives

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\mathbf{x}, t) = 0}$$

which is the Klein-Gordon equation.

We can now compare how each formulation accomplishes the task of ensuring that the expectation value of $\phi(\mathbf{x}, t)$ is a solution to the Klein-Gordon equation. The canonical formulation does this by treating $\phi(\mathbf{x}, t)$ as an operator and assuming that this operator is a solution to the field equation.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\hat{\phi}(\mathbf{x}, t) = 0$$

This is used as the starting point for the derivation of the commonly used expression for $\hat{\phi}(\mathbf{x}, t)$. From this point on, the canonical formulation only requires the use of commutation relations to retain the effects of this derivation.

[2] See Dynin paper.

The Feynman formulation essentially enforces that the expectation value of $\phi(\mathbf{x},t)$ obeys the Klein-Gordon equation through the principle of least action with the Klein-Gordon Lagrangian. The Feynman rules, the cornerstone of practical calculations in modern quantum field theory, have been derived in both the canonical and Feynman formulations, but not in the Schrödinger formulation to our knowledge.[3] The Schrödinger formulation ensures that $\phi(\mathbf{x},t)$ obeys the Klein-Gordon equation by utilizing the Schrödinger equation, which we will address in the next section.

2 Functional Schrödinger Equation

2.1 Klein-Gordon Hamiltonian

The Schrödinger equation governs the time evolution of quantum states.

$$\hat{H}\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle$$

The Schrödinger equation is not simply one choice for the dynamical evolution of states, *it is the only choice.*[4] The Schrödinger equation, up to a proportionality constant in the Hamiltonian, is a direct consequence of the existence of a time-evolution operator in the operator-state formalism. The other dynamical equations, such as the Klein-Gordon equation and the Dirac equation are special cases of the Schrödinger equation. Furthermore, the Schrödinger equation is not a non-relativistic approximation, it is only non-relativistic if a non-relativistic Hamiltonian is inserted. Confusions may arise because the term "Schrödinger equation" is sometimes used to refer to a special case of the equation shown above, meaning that a particular Hamiltonian is inserted.

The main idea behind the Schrödinger formulation of quantum field theory is that the Schrödinger equation enforces the field equation, similar to how in quantum mechanics it enforces Newton's laws in the classical limit. In both quantum mechanics and quantum field theory, Ehrenfest's theorem is the mathematical tool that demonstrates the consequences of the Schrödinger equation on the expectation values of quantities. In order to see the effects of the Schrödinger equation, we must define states for it to act upon. We define $|\Psi\rangle$ to be the state that contains the information in the Schrödinger functional. To extract the Schrödinger functional from this state, we project onto the coordinate basis $\langle\phi|\,\Psi\rangle = \Psi[\phi]$, where $|\phi\rangle$ is defined to be an eigenstate of the coordinate operator, $\hat{\phi}(\mathbf{x},t)\,|\phi\rangle = \phi(\mathbf{x},t)\,|\phi\rangle$. All this is in direct analogy to the case in quantum mechanics under the replacements $\hat{\mathbf{x}} \leftrightarrow \hat{\phi}$, $\mathbf{x} \leftrightarrow \phi$, and $\psi \leftrightarrow \Psi$. We are now quantizing the wave functions of quantum mechanics, hence the term second quantization. The states used in second quantization are not equivalent to normal quantum mechanical states, there is a different type of information being stored in them. First quantized states store information about the probabilities of observing dynamical quantities, whereas second quantized states store information about the probabilities for fields that determine the first quantized wave functions.

These states $|\Psi\rangle$ obey the Schrödinger equation. In order to ensure that the fields ϕ obey the Klein-Gordon equation, we must be sure to choose the correct Hamiltonian for the Schrödinger equation. As we will check in the next section, the proper choice is the Klein-Gordon Hamiltonian (See Appendix for derivation). Therefore, the Schrödinger equation can be written as

$$\boxed{\hat{H}_{KG}\,|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\Psi(t)\rangle}$$

where the Hamiltonian operator is given by

$$\hat{H}_{KG} = \int d^3x\,\left(2c^2\hat{\pi}\hat{\pi}^\dagger + \frac{1}{2}(\boldsymbol{\nabla}\hat{\phi})\cdot(\boldsymbol{\nabla}\hat{\phi}^\dagger) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\hat{\phi}\hat{\phi}^\dagger\right)$$

[3]See Peskin and Schroeder chapters 4 and 9, in particular page 275, which states that chapter 4 uses the canonical formulation and chapter 9 uses the Feynman formulation (which they call the functional integral formulation).

[4]See Sakurai pages 69-72, or for a more explicit version, http://dfcd.net/articles/fieldtheory/schrodinger.pdf

By analogy with the position representation of the momentum operator in quantum mechanics $\hat{p} \doteq -i\hbar\frac{\partial}{\partial x}$, we express the Klein-Gordon Hamiltonian density in the coordinate basis by $\hat{\pi}(\mathbf{x},t) \doteq -i\hbar\frac{\delta}{\delta\phi(\mathbf{x},t)}$ and $\hat{\pi}^\dagger(\mathbf{x},t) \doteq -i\hbar\frac{\delta}{\delta\phi(\mathbf{x},t)}$.

$$\boxed{\hat{\mathcal{H}}_{KG} \doteq -2\hbar^2c^2\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi^*} + \frac{1}{2}(\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi^*) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi\phi^*}$$

In order that all terms in this expression have the same units, we deduce that the dimension of ϕ^2 is energy divided by length.

2.2 Ehrenfest Theorem

Now that we have the Klein-Gordon Hamiltonian we can show that if the functional Schrödinger equation is satisfied, then Ehrenfest's theorem implies that the expectation value of the field obeys the Klein-Gordon equation. So we start by assuming

$$\hat{H}_{KG}\left|\Psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle$$

which gives us the Ehrenfest theorem

$$\frac{d}{dt}\left\langle\hat{A}\right\rangle = \left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle + \frac{1}{i\hbar}\left\langle[\hat{A},\hat{H}]\right\rangle$$

Recall that the Ehrenfest theorem is derived in the Schrödinger picture, so the first term on the right hand side is zero. [5] We will set $\hat{A} \to \hat{\pi}^\dagger(\mathbf{x}') \doteq -i\hbar\frac{\delta}{\delta\phi^*(\mathbf{x}')}$, where the time coordinate is missing because it appears in the state kets in the Schrödinger picture. Before we compute the commutator, it will be helpful to prove a simple lemma. For any differential operator $\hat{D}$ and any operator $\hat{H}$ we have $[\hat{D},\hat{H}] = \hat{D}(\hat{H})$ because

$$[\hat{D},\hat{H}]\psi = (\hat{D}\hat{H} - \hat{H}\hat{D})\psi = \hat{D}(\hat{H}\psi) - \hat{H}(\hat{D}\psi) = (\hat{D}\hat{H})\psi + \hat{H}(\hat{D}\psi) - \hat{H}(\hat{D}\psi) = (\hat{D}\hat{H})\psi$$

With this we compute the commutator in the coordinate representation.

$$[\hat{\pi}^\dagger(\mathbf{x}),\hat{H}_{KG}] \doteq -i\hbar\int d^3x' \left[\frac{\delta}{\delta\phi^*(\mathbf{x})},\hat{\mathcal{H}}_{KG}(\mathbf{x}')\right]$$

The first term drops because functional derivatives commute,

$$= -i\hbar\int d^3x' \left\{\frac{1}{2}\left[\frac{\delta}{\delta\phi^*(\mathbf{x})},\boldsymbol{\nabla}'\phi(\mathbf{x}')\cdot\boldsymbol{\nabla}'\phi^*(\mathbf{x}')\right] + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\left[\frac{\delta}{\delta\phi^*(\mathbf{x})},\phi(\mathbf{x}')\phi^*(\mathbf{x}')\right]\right\}$$

Now using the lemma,

$$= -i\hbar\int d^3x' \left\{\frac{1}{2}\boldsymbol{\nabla}'\phi(\mathbf{x}')\cdot\boldsymbol{\nabla}'\frac{\delta\phi^*(\mathbf{x}')}{\delta\phi^*(\mathbf{x})} + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x}')\frac{\delta\phi^*(\mathbf{x}')}{\delta\phi^*(\mathbf{x})}\right\}$$

$$= -i\hbar\int d^3x' \left\{\frac{1}{2}\boldsymbol{\nabla}'\phi(\mathbf{x}')\cdot\boldsymbol{\nabla}'\delta(\mathbf{x}'-\mathbf{x}) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x})\right\}$$

Using integration by parts in the first term,

$$= -i\hbar\int d^3x' \left\{-\frac{1}{2}\left(\nabla'^2\phi(\mathbf{x}')\right)\delta(\mathbf{x}'-\mathbf{x}) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x}')\delta(\mathbf{x}'-\mathbf{x})\right\}$$

[5] It is only nonzero in cases of explicit time dependence such as when you have a time-varying external electric field.

$$= -i\hbar\left(-\frac{1}{2}\nabla^2\phi(\mathbf{x}) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x})\right)$$

Finally we plug this into the Ehrenfest theorem,

$$\frac{d}{dt}\left\langle\Psi(t)\right|\hat{\pi}^\dagger(\mathbf{x})\left|\Psi(t)\right\rangle = \frac{1}{i\hbar}\left\langle\Psi(t)\right|\left[\hat{\pi}^\dagger(\mathbf{x}),\hat{H}_{KG}\right]\left|\Psi(t)\right\rangle$$

$$= \frac{1}{i\hbar}\left\langle\Psi(t)\right| - i\hbar\left(-\frac{1}{2}\nabla^2\phi(\mathbf{x}) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x})\right)\left|\Psi(t)\right\rangle$$

We must now switch back to the Heisenberg picture,

$$\frac{d}{dt}\left\langle\Psi\right|\hat{\pi}^\dagger(\mathbf{x},t)\left|\Psi\right\rangle = \left\langle\Psi\right|\left(\frac{1}{2}\nabla^2\phi(\mathbf{x},t) - \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x},t)\right)\left|\Psi\right\rangle$$

Using the expression obtained in the appendix $\pi^\dagger(\mathbf{x},t) \doteq \frac{1}{2}\frac{1}{c^2}\dot{\phi}(\mathbf{x},t)$,

$$\frac{d}{dt}\left\langle\Psi\right|\frac{1}{2}\frac{1}{c^2}\dot{\phi}(\mathbf{x},t)\left|\Psi\right\rangle = \left\langle\Psi\right|\left(\frac{1}{2}\nabla^2\phi(\mathbf{x},t) - \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x},t)\right)\left|\Psi\right\rangle$$

$$\left\langle\Psi\right|\frac{1}{2}\frac{1}{c^2}\ddot{\phi}(\mathbf{x},t) - \frac{1}{2}\nabla^2\phi(\mathbf{x},t) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x},t)\left|\Psi\right\rangle = 0$$

Multiplying both sides by a factor of 2 we get,

$$\left\langle\Psi\right|\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\mathbf{x},t)\left|\Psi\right\rangle = 0$$

Or since $|\Psi\rangle$ does not depend on space or time coordinates,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\left\langle\Psi\right|\hat{\phi}(\mathbf{x},t)\left|\Psi\right\rangle = 0$$

where we have converted back to the operator representation from the coordinate representation using the fact that $\hat{\phi}(\mathbf{x},t) \doteq \phi(\mathbf{x},t)$. This result means that the expectation value of the field ϕ obeys the Klein-Gordon equation.

3 Complex Harmonic Oscillators

3.1 Re-expressing the Hamiltonian

The Klein-Gordon Hamiltonian can be re-expressed in a more illuminating way. If we perform some manipulations, we can write it is an integral over complex harmonic oscillator Hamiltonians. The term "complex harmonic oscillator" refers to a quantum oscillator that oscillates in the complex plane rather than just along the real axis. We will see that this leads to a natural interpretation of the field as a three-dimensional continuous lattice of coupled complex harmonic oscillators.

First we define $\hat{\mathcal{H}}'(\mathbf{x})$ to be the non-kinetic part of the Klein-Gordon Hamiltonian density, i.e.

$$\hat{\mathcal{H}}'(\mathbf{x}) \equiv \frac{1}{2}\boldsymbol{\nabla}\phi(\mathbf{x})\cdot\boldsymbol{\nabla}\phi^*(\mathbf{x}) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x})\phi^*(\mathbf{x})$$

Our goal is to convert this into a form that looks like a harmonic oscillator potential, i.e. something that looks like $\frac{1}{2}m^2\omega^2\phi^2$, but in a complexified form. The method is based on Fourier decomposition. We start by inserting

$$\phi(\mathbf{x}) = \int\frac{d^3q}{(2\pi\hbar)^3}e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\tilde{\phi}(\mathbf{q}) \quad \text{where} \quad \tilde{\phi}(\mathbf{q}) = \int d^3x\ e^{-i\mathbf{q}\cdot\mathbf{x}/\hbar}\phi(\mathbf{x})$$

and

$$\phi^*(\mathbf{x}) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{-i\mathbf{q}\cdot\mathbf{x}/\hbar}\tilde{\phi}^*(\mathbf{q}) \quad \text{where} \quad \tilde{\phi}^*(\mathbf{q}) = \int d^3x \; e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\phi^*(\mathbf{x})$$

Note that $\tilde{\phi}^*$ is not the Fourier transform of ϕ^*, it is the complex conjugate of $\tilde{\phi}$. Therefore

$$\boldsymbol{\nabla}\phi(\mathbf{x}) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}(i\mathbf{q}/\hbar)\tilde{\phi}(\mathbf{q})$$

and

$$\boldsymbol{\nabla}\phi^*(\mathbf{x}) = \int \frac{d^3q'}{(2\pi\hbar)^3} e^{-i\mathbf{q}'\cdot\mathbf{x}/\hbar}(-i\mathbf{q}'/\hbar)\tilde{\phi}^*(\mathbf{q}')$$

So

$$\boldsymbol{\nabla}\phi(\mathbf{x})\cdot\boldsymbol{\nabla}\phi^*(\mathbf{x}) = \int \frac{d^3q}{(2\pi\hbar)^3}\frac{d^3q'}{(2\pi\hbar)^3} e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}/\hbar}(\mathbf{q}\cdot\mathbf{q}'/\hbar^2)\tilde{\phi}(\mathbf{q})\tilde{\phi}^*(\mathbf{q}')$$

Finally we have

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi\hbar)^3}\frac{d^3q'}{(2\pi\hbar)^3} e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}/\hbar}(\mathbf{q}\cdot\mathbf{q}'/\hbar^2 + m^2c^2/\hbar^2)\tilde{\phi}(\mathbf{q})\tilde{\phi}^*(\mathbf{q}')$$

Now we need to pull a little trick that relies on the fact that this Hamiltonian density is going to be integrated over all space. It is important to note that the Hamiltonian density is not unique. There are multiple Hamiltonian densities that produce the same Hamiltonian. Furthermore, we have no reason to expect that one Hamiltonian density is more correct than any other, except for the fact that some may be simpler than others. The trick is to notice that integration over all $\mathbf{x}$ will produce a delta function of the form $\delta(\mathbf{q}-\mathbf{q}')$, so we can interchange $\mathbf{q}$ and $\mathbf{q}'$ whenever it is convenient.

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi\hbar)^3}\frac{d^3q'}{(2\pi\hbar)^3} e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}/\hbar}\sqrt{\mathbf{q}^2/\hbar^2 + m^2c^2/\hbar^2}\,\tilde{\phi}(\mathbf{q})\sqrt{\mathbf{q}'^2/\hbar^2 + m^2c^2/\hbar^2}\,\tilde{\phi}^*(\mathbf{q}')$$

Now we define $\omega(\mathbf{q}) \equiv \frac{1}{\hbar}\sqrt{\mathbf{q}^2c^2 + m^2c^4} = c\sqrt{\mathbf{q}^2/\hbar^2 + m^2c^2/\hbar^2}$ so that

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}(\omega(\mathbf{q})/c)\tilde{\phi}(\mathbf{q})\int \frac{d^3q'}{(2\pi\hbar)^3} e^{-i\mathbf{q}'\cdot\mathbf{x}/\hbar}(\omega(\mathbf{q}')/c)\tilde{\phi}^*(\mathbf{q}')$$

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2c^2}\left|\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})\right|^2$$

Finally we multiply and divide by $|\phi(\mathbf{x})|^2$.

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2c^2}\left|\frac{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\tilde{\phi}(\mathbf{q})}\right|^2 |\phi(\mathbf{x})|^2$$

This suggests the definition

$$\boxed{\omega_\phi(\mathbf{x}) \equiv \frac{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})}{\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\tilde{\phi}(\mathbf{q})}}$$

which allows us to write

$$\hat{\mathcal{H}}'(\mathbf{x}) = \frac{1}{2c^2}\left|\omega_\phi(\mathbf{x})\right|^2\left|\phi(\mathbf{x})\right|^2$$

We can see that our Hamiltonian density looks very reminiscent of a simple harmonic oscillator Hamiltonian

$$\boxed{\hat{\mathcal{H}}_{KG}(\mathbf{x}) = 2c^2\hat{\pi}(\mathbf{x})\hat{\pi}^*(\mathbf{x}) + \frac{1}{2c^2}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2}$$

This definition of $\omega_\phi(\mathbf{x})$ is essentially a weighted average of the frequency of oscillation at the point $\mathbf{x}$. It is possible to rewrite $\omega_\phi(\mathbf{x})$ in terms of the convolution operator, which is defined by $(f * g)(x) = \int dx'\ f(x')g(x - x')$.

$$\begin{aligned}
\omega_\phi(\mathbf{x}) &= \frac{1}{\phi(\mathbf{x})} \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \omega(\mathbf{q}) \tilde{\phi}(\mathbf{q}) \\
&= \frac{1}{\phi(\mathbf{x})} \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \omega(\mathbf{q}) \int d^3x'\ e^{-i\mathbf{q}\cdot\mathbf{x}'/\hbar} \phi(\mathbf{x}') \\
&= \frac{1}{\phi(\mathbf{x})} \int d^3x'\ \phi(\mathbf{x}') \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')/\hbar} \omega(\mathbf{q}) \\
&= \frac{1}{\phi(\mathbf{x})} \int d^3x'\ \phi(\mathbf{x}') \tilde{\omega}(\mathbf{x} - \mathbf{x}')
\end{aligned}$$

Therefore

$$\boxed{\omega_\phi(\mathbf{x}) = \frac{(\phi * \tilde{\omega})(\mathbf{x})}{\phi(\mathbf{x})}}$$

where

$$\tilde{\omega}(\mathbf{x}) \equiv \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} \omega(\mathbf{q})$$

Note that $\tilde{\omega}(\mathbf{x})$ is real because after taking the complex conjugate, we can substitute $\mathbf{u} = -\mathbf{q}$, which produces the same integral because $\omega(\mathbf{q})$ is even. By the same method, we can see that $\tilde{\omega}(\mathbf{x}) = \tilde{\omega}(-\mathbf{x})$.

3.2 Physical Interpretation

At this point we can present a physical interpretation of the QFT field as a three-dimensional continuous lattice of coupled complex harmonic oscillators. Let's take a look at the functional Schrödinger equation

$$\hat{H}_{KG} \Psi[\phi, t] = i\hbar \frac{\partial}{\partial t} \Psi[\phi, t]$$

$$\int d^3x\ \left\{ \hat{\mathcal{H}}_{KG}(\mathbf{x}) \Psi[\phi, t] \right\} = i\hbar \frac{\partial}{\partial t} \Psi[\phi, t]$$

This equation suggests that the time derivative breaks up into an integral over all space. The most obvious way to make this happen is to make $\Psi[\phi]$ a continuous product over all space, $\Psi[\phi, t] = \prod_{\mathbf{x}} F_{\mathbf{x}}[\phi, t]$ for some undetermined functional $F_{\mathbf{x}}[\phi, t]$. Now we also note that $\hat{\mathcal{H}}_{KG}(\mathbf{x})$ will commute with any $F_{\mathbf{x}'}[\phi, t]$ when $\mathbf{x}' \neq \mathbf{x}$ because functional derivatives at different points are zero. Therefore the equation looks like

$$\int d^3x\ \left(\hat{\mathcal{H}}_{KG}(\mathbf{x}) F_{\mathbf{x}}[\phi, t] \right) \prod_{\mathbf{x}' \neq \mathbf{x}} F_{\mathbf{x}'}[\phi, t] = -i\hbar \int d^3x\ \frac{\partial}{\partial t} \left(F_{\mathbf{x}}[\phi, t] \right) \prod_{\mathbf{x}' \neq \mathbf{x}} F_{\mathbf{x}'}[\phi, t]$$

By equating the coefficients of the products, this suggests that

$$\hat{\mathcal{H}}_{KG}(\mathbf{x}) F_{\mathbf{x}}[\phi, t] = -i\hbar \frac{\partial}{\partial t} F_{\mathbf{x}}[\phi, t]$$

or in other words that $F_{\mathbf{x}}[\phi, t]$ is a solution to Schrödinger's equation with Hamiltonian $\hat{\mathcal{H}}_{KG}(\mathbf{x})$. When we look back at $\hat{\mathcal{H}}_{KG}(\mathbf{x})$ we see that it is a complex harmonic oscillator type Hamiltonian with coordinate $\phi(\mathbf{x}, t)$. Therefore we are inspired to declare

$$F_{\mathbf{x}}[\phi, t] = \psi_{\mathbf{x}}^{\phi}(\phi(\mathbf{x}, t))$$

and therefore since we set $\Psi[\phi, t] = \prod_{\mathbf{x}} F_{\mathbf{x}}[\phi, t]$,

$$\Psi[\phi, t] = \prod_{\mathbf{x}} \psi_{\mathbf{x}}^{\phi}(\phi(\mathbf{x}, t))$$

This gives us a very intuitive interpretation of the Schrödinger functional $\Psi[\phi, t]$. Essentially, to determine the probability amplitude that the field $\phi(\mathbf{x})$ will manifest upon measurement at given time, we need to determine the probability amplitude that the complex oscillator at point $\mathbf{x}$ takes on the value $\phi(\mathbf{x})$ and then multiply over all points in space according to the multiplicative property of probabilities. This is perhaps the central concept of quantum field theory. It means that the wave functions of particles can be interpreted as being determined by the wave functions of complex oscillators in a field of coupled oscillators. Since continuous products are hard to define, we will rewrite this expression to avoid them

$$\boxed{\Psi[\phi, t] = e^{\ln(\prod_{\mathbf{x}} \psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x},t)))} = e^{\Lambda^{-3}\int d^3x \ln(\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x},t)))}}$$

where Λ is a constant with dimensions of length. This constant may be related to the lattice spacing between the oscillators. Next we will determine the form of ψ in this expression.

4 Ground State

4.1 Complex Harmonic Oscillator Solution

In order to determine the functional form of $\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x}))$, we will simply generate an Ansatz by analogy with the real harmonic oscillator and check that it works. After realizing that the most obvious analogous forms do not work, we are led to try the following.

$$\boxed{\psi_0(\phi(\mathbf{x}), \phi^*(\mathbf{x}')) = e^{-|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2/2\hbar c^2\tilde{\omega}(0)}}$$

Our task will be to show that this satisfies the Schrödinger equation with a Hamiltonian given by the Klein-Gordon Hamiltonian density evalutated at one point. It will help to pre-compute the following functional derivative.

$$\frac{\delta}{\delta\phi(\mathbf{x}')}\left(\omega_\phi(\mathbf{x})\phi(\mathbf{x})\right) = \frac{\delta}{\delta\phi(\mathbf{x}')}(\phi * \tilde{\omega})(\mathbf{x}) = \int d^3x''\ \tilde{\omega}(\mathbf{x}'')\delta^3((\mathbf{x}-\mathbf{x}'')-\mathbf{x}') = \tilde{\omega}(\mathbf{x}-\mathbf{x}')$$

Applying the first functional derivative,

$$\begin{aligned}
\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_0 &= -\frac{1}{2\hbar c^2\tilde{\omega}(0)}\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2\right)\psi_0 \\
&= -\frac{1}{2\hbar c^2\tilde{\omega}(0)}\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\omega^*_\phi(\mathbf{x})\phi^*(\mathbf{x})\right)\omega_\phi(\mathbf{x})\phi(\mathbf{x})\psi_0 \\
&= -\frac{1}{2\hbar c^2}\omega_\phi(\mathbf{x})\phi(\mathbf{x})\psi_0
\end{aligned}$$

Now we apply the second functional derivative

$$\begin{aligned}
\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_0 &= -\frac{1}{2\hbar c^2}\frac{\delta}{\delta\phi(\mathbf{x})}\left(\omega_\phi(\mathbf{x})\phi(\mathbf{x})\psi_0\right) \\
&= -\frac{1}{2\hbar c^2}\left(\frac{\delta}{\delta\phi(\mathbf{x})}\left(\omega_\phi(\mathbf{x})\phi(\mathbf{x})\right)\psi_0 + \omega_\phi(\mathbf{x})\phi(\mathbf{x})\frac{\delta\psi_0}{\delta\phi(\mathbf{x})}\right) \\
&= -\frac{1}{2\hbar c^2}\left(\tilde{\omega}(0)\psi_0 + \omega_\phi(\mathbf{x})\phi(\mathbf{x})\left(-\frac{1}{2\hbar c^2}\omega^*_\phi(\mathbf{x})\phi^*(\mathbf{x})\psi_0\right)\right) \\
&= \left(\frac{1}{4\hbar^2c^4}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2 - \frac{1}{2\hbar c^2}\tilde{\omega}(0)\right)\psi_0
\end{aligned}$$

Multiplying by $-2\hbar^2c^2$ to both sides of this equation yields

$$-2\hbar^2c^2\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_0 = -\frac{1}{2c^2}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2\psi_0 + \hbar\tilde{\omega}(0)\psi_0$$

$$-2\hbar^2c^2\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_0 + \frac{1}{2c^2}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2\psi_0 = u_0\psi_0$$

which is the desired differential equation with an infinite energy density $u_0 = \hbar\tilde{\omega}(0)$.

4.2 Schrödinger Functional Ground State

We can compute the ground state of the Schrödinger functional[6] $\Psi_0[\phi]$ using $\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x})) = e^{-|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2/2\hbar c^2\tilde{\omega}(0)}$

$$\Psi_0[\phi] = \exp\left[\int d^3x\ \ln(\psi^{\phi}_{\mathbf{x}}(\phi(\mathbf{x},t)))\right]$$
$$= \exp\left[-\frac{1}{2\hbar c^2\tilde{\omega}(0)}\int d^3x\ |\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2\right]$$
$$= \exp\left[-\frac{1}{2\hbar c^2\tilde{\omega}(0)}\int d^3x \int \frac{d^3q}{(2\pi\hbar)^3}\int \frac{d^3q'}{(2\pi\hbar)^3} e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}/\hbar}\omega(\mathbf{q})\omega(\mathbf{q}')\tilde{\phi}(\mathbf{q})\tilde{\phi}^*(\mathbf{q}')\right]$$
$$= \exp\left[-\frac{1}{2\hbar c^2\tilde{\omega}(0)}\int \frac{d^3q}{(2\pi\hbar)^3}\int \frac{d^3q'}{(2\pi\hbar)^3}(2\pi\hbar)^3\delta^3(\mathbf{q}-\mathbf{q}')\omega(\mathbf{q})\omega(\mathbf{q}')\tilde{\phi}(\mathbf{q})\tilde{\phi}^*(\mathbf{q}')\right]$$
$$\boxed{\Psi_0[\phi] = \exp\left[-\frac{1}{2\hbar c^2\tilde{\omega}(0)}\int \frac{d^3q}{(2\pi\hbar)^3}\omega^2(\mathbf{q})|\tilde{\phi}(\mathbf{q})|^2\right]}$$

5 Excited States

5.1 Complex Oscillator Excitations

We need to propose a new Ansatz for the excited states. The Ansatz is

$$\boxed{\psi_n(\phi(\mathbf{x}),\phi^*(\mathbf{x})) = \omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x})\psi_0(\phi(\mathbf{x}),\phi^*(\mathbf{x}))}$$

We proceed as before,

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_n = \omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x})\frac{\delta\psi_0}{\delta\phi^*(\mathbf{x})}$$

$$\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_n = \frac{\delta}{\delta\phi(\mathbf{x})}(\omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x}))\frac{\delta\psi_0}{\delta\phi^*(\mathbf{x})} + \omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x})\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta\psi_0}{\delta\phi^*(\mathbf{x})}$$
$$= n\omega^{n-1}_\phi(\mathbf{x})\phi^{n-1}(\mathbf{x})\tilde{\omega}(0)\frac{\delta\psi_0}{\delta\phi^*(\mathbf{x}')} + \omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x})\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta\psi_0}{\delta\phi^*(\mathbf{x})}$$
$$= n\omega^{n-1}_\phi(\mathbf{x})\phi^{n-1}(\mathbf{x})\tilde{\omega}(0)\left(-\frac{1}{2\hbar c^2}\omega_\phi(\mathbf{x})\phi(\mathbf{x})\psi_0\right) + \omega^n_\phi(\mathbf{x})\phi^n(\mathbf{x})\left(\frac{1}{4\hbar^2c^4}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2 - \frac{1}{2\hbar c^2}\tilde{\omega}(0)\right)\psi_0$$
$$= -\frac{1}{2\hbar c^2}n\tilde{\omega}(0)\psi_n + \left(\frac{1}{4\hbar^2c^4}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2 - \frac{1}{2\hbar c^2}\tilde{\omega}(0)\right)\psi_n$$
$$= \left(\frac{1}{4\hbar^2c^4}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2 - \frac{n+1}{2\hbar c^2}\tilde{\omega}(0)\right)\psi_n$$

Again, multiplying by $-2\hbar^2c^2$ gives

$$-2\hbar^2c^2\frac{\delta}{\delta\phi(\mathbf{x})}\frac{\delta}{\delta\phi^*(\mathbf{x})}\psi_n + \frac{1}{2c^2}|\omega_\phi(\mathbf{x})|^2|\phi(\mathbf{x})|^2\psi_n = u_n\psi_n$$

where $u_n = (n+1)\hbar\tilde{\omega}(0)$ is the infinite energy density. Similarly, the complex conjugate of this Ansatz is also a solution.

[6]The result can be compared with equation (10.26) in Hatfield.

5.2 Representing Particles

Suppose we are in a universe that contains just one particle that is localized at a point $\mathbf{x}$ i.e. the particle's wave function is a delta function. How does the theory represent this situation? The only way of encoding information into the Schrödinger functional is through the excitation levels of the complex oscillators at each point. It would not make much sense for a completely localized particle to cause excitations at remote points in space, so we are led to propose that this situation will cause an excitation at $\mathbf{x}$ and we will assume that it goes to the first excited state. What about the case when the particle is not entirely localized? Logically one might guess that the Schrödinger functional would then be a superposition of excitations at various points weighted by the probability of measuring the particle at each point. However, since the Schrödinger functional is a probability amplitude, it should be weighted by something like a probability amplitude, in this case the Klein-Gordon wave function of the particle.[7] If ϕ_1 is the Klein-Gordon wave function, then the Schrödinger functional would be

$$\boxed{\Psi_1^{\phi_1}[\phi] = \int d^3x\ \phi_1(\mathbf{x})\omega_\phi^*(\mathbf{x})\phi^*(\mathbf{x})\Psi_0[\phi]}$$

where $\phi_1(x)$ is the Klein-Gordon wave function of the particle. We have adopted the convention that the complex conjugated form refers to the creation of particles and the un-complex conjugated form refers to the creation of anti-particles. The fact that these two sets of solutions exist with identical spectra is how this theory naturally introduces the idea of anti-particles. [8]

5.3 The Universal Field

What if we are given a Schrödinger functional and we want to know where the particles are inside it? The Schrödinger functional just gives us a probability amplitude for each possible field, but it makes sense that the most probable field would be the field where the particle wave functions are found as expected. We can determine this field, which we call the universal field, by finding the extremum in $\Psi[\phi]$ by the method of Lagrange multipliers with the constraint that the field is normalized. Our normalization condition will be

$$\int d^3x'\ \rho_\phi(\mathbf{x}') = N$$

where

$$\boxed{\rho_\phi(\mathbf{x}) \equiv \frac{i}{2\hbar c^2}(\phi^*\dot{\phi} - \phi\dot{\phi}^*)}$$

is the Klein-Gordon probabilistic number density and N is the number of particles in the field minus the number of anti-particles in the field.[9] We also want to divide out by the ground state wave functional in order to neglect contributions from vacuum fluctuations. The method of Lagrange multipliers tells us to solve

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Psi[\phi]}{\Psi_0[\phi]}\right) = \lambda\frac{\delta}{\delta\phi^*(\mathbf{x})}\int d^3x'\ \frac{i}{2\hbar c^2}(\phi^*\dot{\phi} - \phi\dot{\phi}^*)$$

or

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Psi[\phi]}{\Psi_0[\phi]}\right) = \lambda\frac{i}{2\hbar c^2}\dot{\phi}(\mathbf{x})$$

If we apply this to the case of $\Psi_1^{\phi_1}[\phi]$,

$$\frac{\delta}{\delta\phi^*(\mathbf{x})}\int d^3x'\ \phi_1(\mathbf{x}')\omega_\phi^*(\mathbf{x}')\phi^*(\mathbf{x}') = \lambda\frac{i}{2\hbar c^2}\dot{\phi}(\mathbf{x})$$

[7] See Hatfield equation (2.69).

[8] Using real fields would have a few disadvantages. One is that it would not make as much sense to second quantize complex wave functions with real fields. Another is that this expression would require a more complicated operator to produce the Hermite polynomials of real harmonic oscillator states. That would make the following sections much messier.

[9] The constants in ρ are chosen so as to make N real and dimensionless. We are permitted to multiply by arbitrary constants because any quantity proportional to a conserved quantity is also a conserved quantity.

$$\int d^3x' \; \phi_1(\mathbf{x}')\tilde{\omega}(\mathbf{x}'-\mathbf{x}) = \lambda \frac{i}{2\hbar c^2}\dot{\phi}(\mathbf{x})$$

Therefore

$$\dot{\phi}(\mathbf{x}) = \frac{2\hbar c^2}{i\lambda}(\phi_1 * \tilde{\omega})(\mathbf{x})$$

In order to solve for ϕ in this expression we need to look at the general solution of the Klein-Gordon equation, which is given by

$$\boxed{\phi^{\pm}(\mathbf{x},t) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} e^{\mp i\omega(\mathbf{q})t}\tilde{\phi}(\mathbf{q})}$$

for an arbitrary complex-valued function $\tilde{\phi}(\mathbf{q})$ (See Appendix) This notation means that there are two separate classes of solutions - those with positive energy and those with negative energy, and they do not mix.

For now we are looking at positive energy particles (not anti-particles), so we take the + solution. Taking the time derivative gives

$$\dot{\phi}^{+}(\mathbf{x},t) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} e^{-i\omega(\mathbf{q})t}(-i\omega(\mathbf{q})/\hbar)\tilde{\phi}(\mathbf{q})$$

We can compare this to what we get if we take the convolution with $\tilde{\omega}$.

$$\begin{aligned}
(\phi^{+} * \tilde{\omega})(\mathbf{x},t) &= \int d^3x' \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} e^{-i\omega(\mathbf{q})t}\tilde{\phi}(\mathbf{q}) \int \frac{d^3q'}{(2\pi\hbar)^3} e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{x}')/\hbar}\omega(\mathbf{q}') \\
&= \int \frac{d^3q}{(2\pi\hbar)^3}\frac{d^3q'}{(2\pi\hbar)^3} e^{i\mathbf{q}'\cdot\mathbf{x}/\hbar} e^{-i\omega(\mathbf{q})t}\omega(\mathbf{q}')\tilde{\phi}(\mathbf{q}) \int d^3x' e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}'/\hbar} \\
&= \int \frac{d^3q}{(2\pi\hbar)^3}\frac{d^3q'}{(2\pi\hbar)^3} e^{i\mathbf{q}'\cdot\mathbf{x}/\hbar} e^{-i\omega(\mathbf{q})t}\omega(\mathbf{q}')\tilde{\phi}(\mathbf{q})(2\pi\hbar)^3\delta^3(\mathbf{q}-\mathbf{q}') \\
&= \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} e^{-i\omega(\mathbf{q})t}\omega(\mathbf{q})\tilde{\phi}(\mathbf{q})
\end{aligned}$$

Therefore, for ϕ representing particles

$$\boxed{\dot{\phi} = -\frac{i}{\hbar}\phi * \tilde{\omega} \qquad \text{(for positive energy particles)}}$$

Similarly, $\dot{\phi} = \frac{i}{\hbar}\phi * \tilde{\omega}$ for ϕ representing anti-particles.

Substituting this into the result of the Lagrange multipliers method,

$$-\frac{i}{\hbar}\phi * \tilde{\omega} = \frac{2\hbar c^2}{i\lambda}(\phi_1 * \tilde{\omega})$$

We can then apply the convolution inverse of $\tilde{\omega}$ to both sides to obtain

$$\phi = \frac{2\hbar^2 c^2}{i\lambda}\phi_1$$

Plugging this into the constraint equation and using the normalization of ϕ_1 i.e. $\int d^3x \frac{i}{2\hbar c^2}(\phi_1^*\dot{\phi}_1 - \phi_1\dot{\phi}_1^*) = 1$ gives $\lambda = \frac{2\hbar^2 c^2}{i}$, therefore

$$\boxed{\phi = \phi_1}$$

That is, the most probable field is equivalent to the Klein-Gordon wave function of the particle in the field.

5.4 Two Particles

Since we are dealing with bosons, we should be able to add a second particle to $\Psi_1^{\phi_1}[\phi]$ in the same way we added the first particle to $\Psi_0[\phi]$.

$$\Psi_2^{\phi_1\phi_2}[\phi] = \int d^3x''\ \phi_2(\mathbf{x}'')\phi^*(\mathbf{x}'')\omega_\phi^*(\mathbf{x}'')\Psi_1^{\phi_1}[\phi]$$

$$= \int d^3x''d^3x'\ \phi_1(\mathbf{x}')\phi_2(\mathbf{x}'')\phi^*(\mathbf{x}')\omega_\phi^*(\mathbf{x}')\phi^*(\mathbf{x}'')\omega_\phi^*(\mathbf{x}'')\Phi_0[\phi]$$

The method of Lagrange multipliers gives

$$\lambda\frac{i}{2\hbar c^2}\dot{\phi} = \frac{\delta}{\delta\phi^*(\mathbf{x})}\left(\frac{\Psi_2^{\phi_1\phi_2}[\phi]}{\Psi_0[\phi]}\right)$$

$$= \int d^3x''d^3x'\ \phi_1(\mathbf{x}')\phi_2(\mathbf{x}'')\tilde{\omega}(\mathbf{x}'-\mathbf{x})\phi^*(\mathbf{x}'')\omega_\phi^*(\mathbf{x}'') + \int d^3x''d^3x'\ \phi_1(\mathbf{x}')\phi_2(\mathbf{x}'')\phi^*(\mathbf{x}')\omega_\phi^*(\mathbf{x}')\tilde{\omega}(\mathbf{x}''-\mathbf{x})$$

$$= (\phi_1 * \tilde{\omega})(\mathbf{x})\int d^3x''\ \phi_2(\mathbf{x}'')\phi^*(\mathbf{x}'')\omega_\phi^*(\mathbf{x}'') + (\phi_2 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_1(\mathbf{x}')\phi^*(\mathbf{x}')\omega_\phi^*(\mathbf{x}')$$

$$= (\phi_1 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_2(\mathbf{x}')(\phi^* * \tilde{\omega})(\mathbf{x}') + (\phi_2 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_1(\mathbf{x}')(\phi^* * \tilde{\omega})(\mathbf{x}')$$

In order to find the solution to this equation, we will have to use the orthonormality condition for Klein-Gordon solutions. For particles with positive energy, the orthonormality condition is [10]

$$\int d^3x\ (\phi_a^*\dot{\phi}_b - \dot{\phi}_a^*\phi_b) = -i\delta_{ab}$$

Using the relation derived from the general solution, $\dot{\phi} = -\frac{i}{\hbar}\phi * \tilde{\omega}$ and $\dot{\phi}^* = \frac{i}{\hbar}\phi * \tilde{\omega}$,

$$\int d^3x\ (\phi_a^*(-\frac{i}{\hbar}\phi_b * \tilde{\omega}) - \frac{i}{\hbar}(\phi_a^* * \tilde{\omega})\phi_b) = -i\delta_{ab}$$

$$\int d^3x\ (\phi_a^*(\phi_b * \tilde{\omega}) + (\phi_a^* * \tilde{\omega})\phi_b) = \hbar^2\delta_{ab}$$

$$\int d^3x\ \phi_a^*(\mathbf{x})(\phi_b * \tilde{\omega})(\mathbf{x}) + \int d^3xd^3x'\ \phi_a^*(\mathbf{x}')\tilde{\omega}(\mathbf{x}-\mathbf{x}')\phi_b(\mathbf{x}) = \hbar^2\delta_{ab}$$

$$\int d^3x\ \phi_a^*(\mathbf{x})(\phi_b * \tilde{\omega})(\mathbf{x}) + \int d^3x'\ \phi_a^*(\mathbf{x}')(\phi_b * \tilde{\omega})(\mathbf{x}') = \hbar^2\delta_{ab}$$

$$\int d^3x\ \phi_a^*(\mathbf{x})(\phi_b * \tilde{\omega})(\mathbf{x}) = \frac{1}{2}\hbar^2\delta_{ab}$$

Now we can insert our Ansatz for the universal field $\phi = \phi_1 + \phi_2$.

$$\lambda\frac{i}{2\hbar c^2}(\dot{\phi}_1 + \dot{\phi}_2) = (\phi_1 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_2(\mathbf{x}')((\phi_1^* + \phi_2^*) * \tilde{\omega})(\mathbf{x}') + (\phi_2 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_1(\mathbf{x}')((\phi_1^* + \phi_2^*) * \tilde{\omega})(\mathbf{x}')$$

$$= (\phi_1 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_2(\mathbf{x}')(\phi_2^* * \tilde{\omega})(\mathbf{x}') + (\phi_2 * \tilde{\omega})(\mathbf{x})\int d^3x'\ \phi_1(\mathbf{x}')(\phi_1^* * \tilde{\omega})(\mathbf{x}')$$

[10] See Practical Quantum Electrodynamics equation (4.48). We contacted the author of this book and he told us that he did not have a proof for this expression, so it may be wrong. But even if it is wrong, the result would be similar. You would just have to replace the integrals with unknown time-dependent constants.

$$= \frac{1}{2}\hbar^2(\phi_1 * \tilde{\omega})(\mathbf{x}) + \frac{1}{2}\hbar^2(\phi_2 * \tilde{\omega})(\mathbf{x})$$

Therefore we have

$$-\frac{i}{\hbar}\frac{i\lambda}{2\hbar c^2}(\phi_1 * \tilde{\omega} + \phi_2 * \tilde{\omega}) = \frac{1}{2}\hbar^2(\phi_1 * \tilde{\omega}) + \frac{1}{2}\hbar^2(\phi_2 * \tilde{\omega})$$

which is true for the appropriate choice of λ, so the Ansatz works: the universal field for a two particle state is given by the sum of the Klein-Gordon wave functions for the two particles.

$$\boxed{\phi = \phi_1 + \phi_2}$$

5.5 Indistinguishability

The fact that the universal field is the sum of the Klein-Gordon wave functions of the particles in the field means that there is a loss of information that occurs after adding particles to a field. Even though we might try to keep track of each particle's wave function individually, the universe only keeps track of the sum of the wave functions within each indistinguishability class. To get a picture of what this loss of information means, consider what happens when two identical particles collide, or more precisely when two localized wave functions pass through the same region. This can be visualized more simply by imagining two ripples traveling toward each other on one rope. When the ripples collide, do they reflect off each other or pass through each other unaffected? There is no answer to this question; the two cases are indistinguishable.[11] The same concept applies to the collision of particles. If we were to treat each particle as if it could be tagged and followed, then we would be over-counting the number of possible physical processes. This is why we must account for indistinguishability when working within a particle-based interpretation of physics.

6 What about fermions?

This paper constructed a model based on a three-dimensional lattice of continuously coupled harmonic oscillators. Each oscillator evolves according to a Hamiltonian given by the complex Klein-Gordon Hamiltonian density evaluated at one point. This Hamiltonian is identical to the Hamiltonian of a two-dimensional harmonic oscillator, which suggests that there is something like a two-dimensional plane at each point in space. These planes may be purely mathematical, or they may represent some as yet unobserved physical dimensions of reality. So is there any room for fermions in this model? There may be if we make a slight modification. We will propose one option that is not necessarily correct, but demonstrates how it might be possible to accommodate fermions. Suppose that instead of a plane at each point in space we have a torus i.e. a square whose edges wrap around to the other side. This would be equivalent to a periodic potential, and thus it would only support a finite number of bound states. Picture a ball bearing rolling on the inside of a torus oriented with its axis parallel with the ground. If the ball had a low kinetic energy, it would just oscillate like a pendulum bob. But if it had high enough kinetic energy, it would loop all the way around the perimeter of the torus. This would correspond to an unbound state. The key difference between bound and unbound states is that bound states have an expectation value of momentum of zero, whereas unbound states must pick a direction for their orbits. Quantum mechanically, a jump from a bound state to an unbound state would require a change in momentum. If there is no mechanism for momentum transfer between toruses, then unbound states would be forbidden. So given the appropriate size for the toruses, it could be arranged that each point in space only supports zero or one particles, just as in the case of fermions.

[11] This example comes from Teller's book.

7 Appendix A: General Solution of the Klein-Gordon Equation

The Klein-Gordon equation is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi(\mathbf{x},t) = 0$$

We can find the general solution by inserting the Fourier decomposition of $\phi(\mathbf{x},t)$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{\phi}(\mathbf{q},t) = 0$$

$$\int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\left(\frac{1}{c^2}\ddot{\tilde{\phi}}(\mathbf{q},t) + \frac{q^2}{\hbar^2}\tilde{\phi}(\mathbf{q},t) + \frac{m^2c^2}{\hbar^2}\tilde{\phi}(\mathbf{q},t)\right) = 0$$

This is solved by any function $\tilde{\phi}(\mathbf{q},t)$ that satisfies

$$\ddot{\tilde{\phi}}(\mathbf{q},t) = -\omega^2(\mathbf{q})\tilde{\phi}(\mathbf{q})$$

or

$$\tilde{\phi}(\mathbf{q},t) = e^{\pm i\omega(\mathbf{q})t}\tilde{\phi}(\mathbf{q})$$

Therefore the general solution is

$$\phi^{\pm}(\mathbf{x},t) = \int \frac{d^3q}{(2\pi\hbar)^3} e^{i\mathbf{q}\cdot\mathbf{x}/\hbar} e^{\mp i\omega(\mathbf{q})t}\tilde{\phi}(\mathbf{q})$$

where $\tilde{\phi}(\mathbf{q})$ is an arbitrary complex-valued function.

8 Appendix B: Klein-Gordon Hamiltonian

We start by multiplying the Klein-Gordon equation by $\delta\phi^*$ and integrating over all space and time.

$$\int d^3x\ dt\ \left[\delta\phi^*\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi\right] = 0$$

Integrating by parts in the first and second terms,

$$\int d^3x\ dt\ \left[-\frac{1}{c^2}\frac{\partial\delta\phi^*}{\partial t}\frac{\partial\phi}{\partial t} + \frac{\partial\delta\phi^*}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\delta\phi^*}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial\delta\phi^*}{\partial z}\frac{\partial\phi}{\partial z} + \frac{m^2c^2}{\hbar^2}\phi\delta\phi^*\right] = 0$$

$$\int d^3x\ dt\ \left[-\frac{1}{c^2}\delta\left(\frac{\partial\phi^*}{\partial t}\right)\frac{\partial\phi}{\partial t} + \delta\left(\frac{\partial\phi^*}{\partial x}\right)\frac{\partial\phi}{\partial x} + \delta\left(\frac{\partial\phi^*}{\partial y}\right)\frac{\partial\phi}{\partial y} + \delta\left(\frac{\partial\phi^*}{\partial z}\right)\frac{\partial\phi}{\partial z} + \frac{m^2c^2}{\hbar^2}\phi\delta\phi^*\right] = 0$$

Now we add the same equation with ϕ and ϕ^* swapped and divide by 2.

$$\frac{1}{2}\int d^3x\ dt\ \left[-\frac{1}{c^2}\delta\left(\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t}\right) + \delta\left(\frac{\partial\phi}{\partial x}\frac{\partial\phi^*}{\partial x}\right) + \delta\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi^*}{\partial y}\right) + \delta\left(\frac{\partial\phi}{\partial z}\frac{\partial\phi^*}{\partial z}\right) + \frac{m^2c^2}{\hbar^2}\delta\left(\phi\phi^*\right)\right] = 0$$

$$\delta\int d^3x\ dt\ \frac{1}{2}\left[-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t} + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi^*) + \frac{m^2c^2}{\hbar^2}\phi\phi^*\right] = 0$$

The principle of least action says that $\delta S = 0$, where $S = \int dt\ L$ and $L = \int d^3x\ \mathcal{L}$, and to get the proper Lagrangian density we multiply both sides by -1.

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t} - (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi^*) - \frac{m^2c^2}{\hbar^2}\phi\phi^*\right]$$

Now that we have the Lagrangian density, we can use the canonical change of variables to convert it into a Hamiltonian density. In classical mechanics we had the formula

$$H = \sum_k p_k \dot{q}_k - L$$

Now the values of $\phi(\mathbf{x}, t)$ are the coordinates so we must convert the sum to an integral and then define $\pi(\mathbf{x}, t)$ to be the conjugate momenta.

$$H = \int d^3x \left[\pi(\mathbf{x})\dot{\phi}(\mathbf{x}) + \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x})\right] - L$$

$$\int d^3x\ \mathcal{H} = \int d^3x\ \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x}) + \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x}) - \int d^3x\ \mathcal{L}$$

$$\mathcal{H} = \pi(\mathbf{x})\dot{\phi}(\mathbf{x}) + \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x}) - \mathcal{L}$$

This is not the final result because in the Hamiltonian formalism, there are no explicit time derivatives because they are replaced with the expression for momentum. In classical mechanics we had $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$. Now we have the functional derivative

$$\pi(\mathbf{x}) = \frac{\delta L}{\delta \dot{\phi}(\mathbf{x})}$$

Instead of solving this specifically, we can prove the following lemma.

$$\frac{\delta L[\phi(\mathbf{x})]}{\delta \phi(\mathbf{y})} = \frac{\partial \mathcal{L}}{\partial \phi}(\mathbf{y})$$

By the definition of functional differentiation,

$$\frac{\delta L[\phi(\mathbf{x})]}{\delta \phi(\mathbf{y})} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^3x\ \left\{\mathcal{L}[\phi(\mathbf{x}) + \epsilon\delta(\mathbf{x}-\mathbf{y})] - \mathcal{L}[\phi(\mathbf{x})]\right\}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^3x\ \left\{\mathcal{L}[\phi(\mathbf{x})] + \epsilon\delta(\mathbf{x}-\mathbf{y})\frac{\partial \mathcal{L}}{\partial \phi}(\phi(\mathbf{x})) + \mathcal{O}(\epsilon^2) - \mathcal{L}[\phi(\mathbf{x})]\right\}$$

$$= \int d^3x\ \delta(\mathbf{x}-\mathbf{y})\frac{\partial \mathcal{L}}{\partial \phi}(\phi(\mathbf{x}))$$

$$= \frac{\partial \mathcal{L}}{\partial \phi}(\phi(\mathbf{y}))$$

So we now have

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x})} = \frac{1}{2}\frac{1}{c^2}\dot{\phi}^*(\mathbf{x}, t)$$

and similarly,

$$\pi^*(\mathbf{x}, t) = \frac{1}{2}\frac{1}{c^2}\dot{\phi}(\mathbf{x}, t)$$

Therefore

$$\mathcal{H} = \pi(\mathbf{x})\dot{\phi}(\mathbf{x}) + \pi^*(\mathbf{x})\dot{\phi}^*(\mathbf{x}) - \mathcal{L}$$

$$= 4c^2\pi(\mathbf{x}, t)\pi^*(\mathbf{x}, t) - \mathcal{L}$$

$$= 4c^2\pi(\mathbf{x}, t)\pi^*(\mathbf{x}, t) - \frac{1}{2}\left[4c^2\pi(\mathbf{x}, t)\pi^*(\mathbf{x}, t) - (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi^*) - \frac{m^2c^2}{\hbar^2}\phi\phi^*\right]$$

Finally we obtain

$$\mathcal{H}_{KG} = 2c^2\pi(\mathbf{x}, t)\pi^*(\mathbf{x}, t) + \frac{1}{2}(\boldsymbol{\nabla}\phi(\mathbf{x}, t))\cdot(\boldsymbol{\nabla}\phi^*(\mathbf{x}, t)) + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\phi(\mathbf{x}, t)\phi^*(\mathbf{x}, t)$$

9 Appendix C: Complex Partial Differentiation

Let f be a function of two independent complex parameters z_1 and z_2. We can reparameterize f to express it in terms of just one complex variable and its complex conjugate by defining $z = z_1 + iz_2$ and $z^* = z_1 - iz_2$. This allows us to write $f(z_1, z_2) = f(z, z^*)$. The partial derivative of f with respect to z holds z^* constant on the approach to the limit.

$$0 = \Delta z^* = (z_1 + \Delta z_1) - i(z_2 + \Delta z_2) - (z_1 - iz_2) = \Delta z_1 - i\Delta z_2$$

Therefore, the partial derivative is

$$\frac{\partial f(z,z^*)}{\partial z} = \lim_{\substack{\Delta z \to 0 \\ \Delta z^* = 0}} \frac{f(z+\Delta z, z^*) - f(z,z^*)}{\Delta z}$$

$$= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2}$$

$$= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2 + \Delta z_2) + f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2}$$

$$= \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1 + \Delta z_1, z_2 + \Delta z_2) - f(z_1, z_2 + \Delta z_2)}{\Delta z_1 + i\Delta z_2} + \lim_{\substack{\Delta z_1, z_2 \to 0 \\ \Delta z_1 - i\Delta z_2 = 0}} \frac{f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{\Delta z_1 + i\Delta z_2}$$

$$= \lim_{\Delta z_1 \to 0} \frac{f(z_1 + \Delta z_1, z_2 - i\Delta z_1) - f(z_1, z_2 - i\Delta z_1)}{\Delta z_1 + \Delta z_1} + \lim_{\Delta z_2 \to 0} \frac{f(z_1, z_2 + \Delta z_2) - f(z_1, z_2)}{i\Delta z_2 + i\Delta z_2}$$

$$= \frac{1}{2}\frac{\partial f}{\partial z_1} + \frac{1}{2i}\frac{\partial f}{\partial z_2}$$

If we now restrict z_1 and z_2 to be real, they become the real and imaginary parts of z, which we write as x and y respectively. Therefore, if $z = x + iy$ for $x, y \in \mathbb{R}$,

$$\frac{\partial f(z,z^*)}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

Similarly,

$$\frac{\partial f(z,z^*)}{\partial z^*} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$$

These partial derivatives can be evaluated by treating z and z^* as if they were completely independent parameters, which explains the reason for treating ϕ and ϕ^* as independent fields in quantum field theory.

10 References

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