8. *Statistical Mechanics and Thermodynamics* (Spring 2007)

Consider a large number of photons, in thermal equilibrium at temperature T, inside a volume V. The photons are in vacuum and can have two possible directions of polarization.

(a) Let $u(\omega, T)\, d\omega$ be the mean energy per unit volume of photons in the frequency range between ω and $\omega + d\omega$. Find $u(\omega, T)$.

(b) What is the temperature dependence of the total energy density u_0?

Note: The constants of proportionality will involve a definite integral which you need not solve explicitly – find temperature dependence only.

a) $$u_0 = \frac{1}{V}\langle U\rangle = \frac{1}{V}\int_0^\infty \epsilon\, f(\epsilon)\,\rho(\epsilon)\, d\epsilon \qquad \epsilon = pc = \hbar\omega, \quad f(\epsilon) = \frac{1}{e^{\beta\epsilon}-1}$$

$$\rho = \frac{d\Omega}{d\epsilon} = 2\,\frac{\frac{1}{h^3}\, dV_{phase}}{d\epsilon}$$ (where the 2 comes from polarization multiplicity)

$$dV_{phase} = \left(\int d^3r\right)\left(\int p^2\, d\Omega_p\right) dp = V\, 4\pi p^2\, dp = V\, 4\pi \left(\frac{\hbar}{c}\right)^3 \omega^2\, d\omega \qquad \text{since } p = \frac{\hbar}{c}\omega$$

$$\Rightarrow \rho = 2\cdot 4\pi V \left(\frac{1}{2\pi c}\right)^3 \omega^2 \frac{d\omega}{d\epsilon} = \frac{V}{\pi^2 c^3}\,\omega^2 \frac{d\omega}{d\epsilon}$$

$$\Rightarrow u_0 = \frac{1}{V}\int_0^\infty \hbar\omega \left(\frac{1}{e^{\beta\epsilon}-1}\right) \frac{V}{\pi^2 c^3}\,\omega^2 \frac{d\omega}{d\epsilon}\, d\epsilon$$

$$= \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \int_0^\infty u(\omega, T)\, d\omega$$

$$\Rightarrow u(\omega, T) = \frac{\hbar}{\pi^2 c^3}\,\frac{\omega^3}{e^{\hbar\omega/kT}-1} \qquad \text{(Planck's law)}$$

b) $$u_0 \sim \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega \sim \left(\frac{1}{\beta}\right)^4 \int_0^\infty \frac{x^3\, dx}{e^x - 1} \qquad \text{where } x = \beta\hbar\omega \quad \text{so } \omega \sim \frac{1}{\beta}x$$

$$\sim T^4 \qquad (\sim \text{Stefan-Boltzmann Law})$$