3. *Quantum Mechanics* (Spring 2007)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L}+\mathbf{s}$. Conventionally, the states are labelled $|n,l,j,m_j\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

In parts (a) and (d) you may ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $|n,l,j,m_j\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m_j?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_+|3,1,\frac{3}{2},-\frac{1}{2}\rangle = ?$ $\quad J_+ \to T^1_1 \quad \Delta m = 2 \neq 1 \quad = \emptyset$

(ii) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_+|3,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ $\quad \sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)} = \sqrt{15/4 - 3/4} = \sqrt{3}$

(iii) $\langle 2,1,\frac{1}{2},-\frac{1}{2}|\mathbf{L}^2|2,1,\frac{1}{2},-\frac{1}{2}\rangle = ?$ $\quad \vec{L}^2 \to T^0_0 \quad 1(1+1) = 2$

(iv) $\langle 3,2,\frac{3}{2},-\frac{1}{2}|\mathbf{J}^2|3,2,\frac{3}{2},-\frac{1}{2}\rangle = ?$ $\quad \vec{J}^2 \to T^0_0 \quad \frac{3}{2}(\frac{3}{2}+1) = 15/4$

(v) $\langle 3,1,\frac{3}{2},\frac{3}{2}|J_z|3,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ $\quad J_z \to T^0_1 \quad \Delta m = 1 \neq 0 \quad = \emptyset$

(d) What are

(i) $\langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = ?$ $\quad p_z \to T^0_1 \quad \Delta m = 1 \neq 0 \quad = \emptyset$

(ii) $\langle 1,0,\frac{1}{2},\frac{1}{2}|p_i p_j|1,0,\frac{1}{2},\frac{1}{2}\rangle = ?$

a) $H: \; -\frac{1}{2}\alpha^2 mc^2 \frac{1}{n^2} \qquad \vec{L}^2: \; l(l+1) \qquad \vec{J}^2: \; j(j+1) \qquad J_z: \; m_j$

b) $n \in \mathbb{Z}^+ = \{1,2,3,\ldots\} \qquad l \in \{0,1,2,\ldots,n-1\} \qquad s = \frac{1}{2}$

$j \in \{|l-s|, |l-s|+1, \ldots, l+s-1, l+s\} \qquad m_j \in \{-j, -j+1, \ldots, j-1, j\}$

c) Wigner-Eckart selection rules: $\langle \alpha', j', m' | T^q_k | \alpha, j, m\rangle = 0$ unless $\Delta m = q$ and $|\Delta j| \leq k \leq \Sigma j$

d) ii) $p_i p_j = \underbrace{\frac{1}{3}\vec{p}^2\delta_{ij}}_{T^0_0} + \frac{1}{2}\underbrace{(\cancel{p_ip_j - p_jp_i})}_{[p_i,p_j]=0} + \underbrace{\left[\frac{1}{2}(p_ip_j + p_jp_i) - \frac{1}{3}\vec{p}^2\delta_{ij}\right]}_{\sum_q T^q_2} \qquad k=2 \not\leq \Sigma j = 1$

$\langle p_ip_j\rangle = \frac{1}{3}\langle \vec{p}^2\rangle\delta_{ij} = \frac{1}{3}2m\langle T\rangle\delta_{ij} = -\frac{2m}{3}\langle E\rangle\delta_{ij}$ by virial theorem

$= +\frac{1}{3}\alpha^2 m^2 c^2 \frac{1}{n^2}\delta_{ij}\Big|_{n=1} = \frac{1}{3}\alpha^2 m^2 c^2 \delta_{ij}$