3. *Quantum Mechanics* (Spring 2006)

A hydrogen atom is placed in a constant weak electric field of strength $\mathcal{E}$. Ignoring spin, what are the energies of the $n=1$ and $n=2$ levels including effects to first order in $\mathcal{E}$ (but ignoring second order effects)?

Note: You may want to use some of the following:

Radial Wave Functions $R_{nl}(r)$ (a is the Bohr radius):

$$R_{10}(r) = \frac{1}{a^{3/2}} 2e^{-r/a} \qquad R_{21}(r) = \frac{1}{a^{3/2}}\frac{1}{2\sqrt{6}}\frac{r}{a}e^{-r/2a}$$
$$R_{20}(r) = \frac{1}{a^{3/2}}\frac{1}{\sqrt{2}}\left(1-\frac{r}{2a}\right)e^{-r/2a}$$

Spherical Harmonics $Y_l^m(\theta,\phi)$:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

An integral:

$$\int_0^\infty x^n e^{-x/a}dx = a^{n+1}n!$$

Unperturbed energies: $E_n^{(0)} = -\frac{mK^2e^4}{2\hbar^2n^2}$, degenerate (in l,m) for $n \geq 2$

Perturbation: $H' = -\vec{p}\cdot\vec{\mathcal{E}} = -q\vec{r}\cdot\vec{\mathcal{E}} = e\mathcal{E}z$ given $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$

$n=1$ ($l=m=0$): (nondegenerate perturbation theory)

$(\Delta E)_1^{(1)} = \langle\psi_{100}|H'|\psi_{100}\rangle$, but H' (i.e., z) is an odd-parity operator, which only "connects" states of different parity

$\Rightarrow (\Delta E)_1^{(1)} = 0$

$n=2$ ($|200\rangle, |210\rangle, |21\pm1\rangle$): (degenerate perturbation theory, four-fold)
(rename $\hookrightarrow |0\rangle$ $\hookrightarrow |1\rangle$ $\hookrightarrow |\pm\rangle$)

$$H' \doteq \begin{pmatrix} \langle 0|H'|0\rangle & \langle 0|H'|1\rangle & \langle 0|H'|+\rangle & \langle 0|H'|-\rangle \\ \langle 1|H'|0\rangle & \langle 1|H'|1\rangle & \langle 1|H'|+\rangle & \langle 1|H'|-\rangle \\ \langle +|H'|0\rangle & \langle +|H'|1\rangle & \langle +|H'|+\rangle & \langle +|H'|-\rangle \\ \langle -|H'|0\rangle & \langle -|H'|1\rangle & \langle -|H'|+\rangle & \langle -|H'|-\rangle \end{pmatrix}$$

the Wigner-Eckart selection rules tell us, since z is the zero-component of a rank one spherical tensor, that the non-zero matrix elements have $\Delta m = 0$, $|\Delta l| \leq 1 \leq 2l$ (remember also, H' doesn't connect a state to itself)

$$= \begin{pmatrix} 0 & c & 0 & 0 \\ c^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow (\Delta E)_\pm^{(1)} = 0$$

$c = \langle 0|H'|1\rangle$ where $H' = e\mathcal{E}r\cos\theta = e\mathcal{E}\sqrt{\frac{4\pi}{3}}\,r\,Y_1^0$ given $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$

$$= e\mathcal{E}\sqrt{\frac{4\pi}{3}}\int_0^\infty a^{-3/2}\frac{1}{\sqrt{2}}\left(1-\frac{r}{2a}\right)e^{-r/2a}\,a^{-3/2}\frac{1}{2\sqrt{6}}\frac{r}{a}e^{-r/2a}\,r^3dr\int\frac{1}{\sqrt{4\pi}}Y_1^0Y_1^0\,d\Omega$$

$$= e\mathcal{E}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{6}}a^{-3}a^{-1}\int_0^\infty\left(r^4-\frac{r^5}{2a}\right)e^{-r/a}dr = \frac{1}{12}e\mathcal{E}a^{-4}\left[a^5 4! - \frac{1}{2a}a^6 5!\right]$$

$$= e\mathcal{E}a[2-5] = -3e\mathcal{E}a \qquad \text{and} \quad c^* = c$$

3. Quantum Mechanics (Spring 2006)

(continued)

We want the remaining two eigenvalues of H', so we can simply examine the upper left quarter of the matrix:

$$\begin{pmatrix} 0 & c \\ c^* & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = c\,\sigma_x \quad \Rightarrow \quad |\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

where $\sigma_x |\psi_\pm\rangle = \pm |\psi_\pm\rangle$

$\Rightarrow$ eigenvalues $\pm c$ for $|\psi_\pm\rangle$

Summarizing: $E^{(1)}_{|\psi\rangle} = E^{(0)}_n + (\Delta E)^{(1)}_{|\psi\rangle}$, so

$n=1$ $\quad |100\rangle \qquad E^{(1)}_{100} = -\dfrac{mK^2e^4}{2\hbar^2}$

$n=2$ $\quad |\psi_+\rangle = \dfrac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right) \qquad E^{(1)}_{|\psi_+\rangle} = -\dfrac{mK^2e^4}{8\hbar^2} - 3e\mathcal{E}a$

$|\psi_-\rangle = \dfrac{1}{\sqrt{2}}\left(|200\rangle - |210\rangle\right) \qquad E^{(1)}_{|\psi_-\rangle} = -\dfrac{mK^2e^4}{8\hbar^2} + 3e\mathcal{E}a$

$|21\,{+1}\rangle \qquad E^{(1)}_{21+1} = -\dfrac{mK^2e^4}{8\hbar^2}$

$|21\,{-1}\rangle \qquad E^{(1)}_{21-1} = -\dfrac{mK^2e^4}{8\hbar^2}$