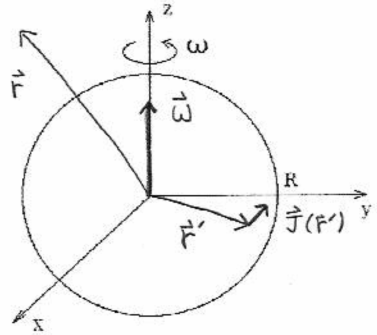
14. *Electricity and Magnetism* (Fall 2006)

Consider a rotating sphere with radius R. A charge Q is distributed homogeneously over the sphere. The sphere rotates counter-clockwise around the z-axis with angular velocity ω. (See figure below.)



(a) Find the charge density ρ and the current density $\mathbf{j}$ in terms of delta functions. Show that $\boldsymbol{\nabla}\cdot\mathbf{j} = 0$.

(b) Find the vector potential $\mathbf{A}(\mathbf{x})$ in the Coulomb gauge ($\boldsymbol{\nabla}\cdot\mathbf{A} = 0$).

Hint: To do the integral it is advantageous to choose $\mathbf{r} = r\mathbf{e}_z$ and choose $\boldsymbol{\omega}$ to be arbitrary.

(c) Calculate the magnetic field $\mathbf{B}$ from the vector potential $\mathbf{A}$.

a) $\rho(\vec{r}) = \frac{Q}{4\pi R^2}\,\delta(r-R)$ $\quad\quad$ $\int_{\mathbb{R}^3} \rho(\vec{r})\,dv = \frac{Q}{4\pi R^2}\int \delta(r-R)\,r^2 dr\, d\Omega = \frac{Q}{4\pi R^2} R^2 4\pi$
$= Q$ ✓

$\vec{J}(\vec{r}) = \rho(\vec{r})\,\vec{v}(\vec{r})$ where $\vec{v}(\vec{r}) = v_\phi\hat{\phi} = s\omega\hat{\phi} = R\sin\theta\,\omega\,\hat{\phi}$
$= \rho(\vec{r})\,R\omega\sin\theta\,\hat{\phi} = \frac{Q\omega}{4\pi R}\,\delta(r-R)\sin\theta\,\hat{\phi}$

$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2 S\theta}\left\langle \partial_r, \frac{1}{r}\partial_\theta, \frac{1}{rS\theta}\partial_\phi\right\rangle_s \cdot \left(r^2 S\theta\,\langle 0, 0, J_\phi\rangle_s\right) = 0$ since J_ϕ is independent of ϕ ✓

b) $\vec{A}(\vec{r}) = K_m \int_{\mathbb{R}^3} \frac{\vec{J}(\vec{r}\,')\,dv'}{|\vec{r}-\vec{r}\,'|} = K_m \frac{Q\omega}{4\pi R}\int \frac{\delta(r'-R)\sin\theta'\,\hat{\phi}'\,r'^2 dr'\,d\Omega'}{|\vec{r}-\vec{r}\,'|}$

$= K_m \frac{Q\omega}{4\pi R} R^2 \int \frac{\sin\theta'\,\langle -\sin\phi', \cos\phi', 0\rangle_d\, d\Omega'}{|\vec{r} - \vec{R}(\theta',\phi')|}$ $\quad\quad$ $C\sin\theta\cos\phi = \frac{1}{2}\left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right]$
$C\sin\theta\sin\phi = \frac{1}{2i}\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right]$

$= K_m \frac{Q\omega R}{\cancel{4\pi}} \int \left\langle -\frac{1}{2iC}\left[Y_{11}(\theta',\phi') - Y_{1,-1}(\theta',\phi')\right], \frac{1}{2C}\left[Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')\right], 0\right\rangle_d \sum_{lm} \frac{\cancel{4\pi}}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta',\phi')\,d\Omega'$

$= K_m Q\omega R \frac{1}{2(1)+1}\frac{r_<}{r_>^2}\left\langle -\frac{1}{2iC}\left[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi')\right], \frac{1}{2C}\left[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)\right], 0\right\rangle_d$ $\quad\quad$ $r_<, r_> \in \{r, R\}$

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b) (continued)

$$= \frac{1}{3} K_m Q\omega R \frac{r_<}{r_>^2} \sin\theta \langle -\sin\phi, \cos\phi, 0\rangle_d$$

$$= \frac{1}{3} K_m Q\omega R \frac{r_<}{r_>^2} \sin\theta \, \hat{\phi}$$

$$\vec{A}(\vec{r}) = \frac{1}{3} K_m Q\omega \begin{Bmatrix} r/R, & r<R \\ R^2/r^2, & r>R \end{Bmatrix} \sin\theta \, \hat{\phi}$$

$\vec{\nabla}\cdot\vec{A} = 0$ since $\vec{A} = A_\phi \hat{\phi}$ is not ϕ-dependent (like $\vec{J}$) ✓

c) $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}$

$$= \frac{1}{r^2 S\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & rS\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \cancel{A_r} & \cancel{rA_\theta} & rS\theta A_\phi \end{vmatrix} = \frac{1}{r^2 S\theta} \langle \partial_\theta [rS\theta A_\phi], r(-\partial_r [rS\theta A_\phi]), 0\rangle_s$$

$$= \frac{1}{3} K_m Q\omega \frac{1}{r^2 S\theta} \left\langle \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix} \partial_\theta (S^2\theta), -r S^2\theta \, \partial_r \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix}, 0 \right\rangle_s$$

$$= \frac{1}{3} K_m Q\omega \frac{1}{r^2 \cancel{S\theta}} \left\langle \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix} 2\cancel{S\theta} C\theta, \begin{Bmatrix} -2r^2/R \\ R^2/r \end{Bmatrix} S^2\theta, 0 \right\rangle_s$$

$$= \frac{1}{3} K_m Q\omega \begin{Bmatrix} 1/R, & r<R \\ R^2/r^3, & r>R \end{Bmatrix} \left\langle 2\cos\theta, \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} \sin\theta, 0 \right\rangle_s$$

Check: $\vec{\nabla}\cdot\vec{B} = \frac{1}{r^2 S\theta} \langle \partial_r, \frac{1}{r}\partial_\theta, \cancel{\frac{1}{rS\theta}\partial_\phi} \rangle_s \cdot \left(r^2 S\theta \langle B_r, B_\theta, 0\rangle_s \right)$

$$= \frac{1}{3} K_m Q\omega \frac{1}{r^2 S\theta} \left[\partial_r \left(r^2 S\theta \begin{Bmatrix} 1/R \\ R^2/r^3 \end{Bmatrix} 2C\theta \right) + \frac{1}{r} \partial_\theta \left(r^2 S\theta \begin{Bmatrix} -2/R \\ R^2/r^3 \end{Bmatrix} S\theta \right) \right]$$

$$= \frac{1}{3} K_m Q\omega \frac{1}{r^2 S\theta} \left[\begin{Bmatrix} 2r/R \\ -R^2/r^2 \end{Bmatrix} 2S\theta C\theta + \begin{Bmatrix} -2r/R \\ R^2/r^2 \end{Bmatrix} 2S\theta C\theta \right]$$

$$= 0$$ ✓