10. *Electricity and Magnetism* (Fall 2006)

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

(a) What is the total charge of this system? Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

(b) Identify any point charges in this system and give their location and charge.

(c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

a) Examine a sphere of radius R and the total charge enclosed Q_R:

$$Q_R = \int_{V_R} \rho \, dv = \varepsilon_0 \int_{V_R} (\vec{\nabla}\cdot\vec{E})\, dv = \varepsilon_0 \int_{S_R} \vec{E}\cdot d\vec{a} = -\varepsilon_0 \int_{S_R} \vec{\nabla}\Phi \cdot d\vec{a}$$

$$\vec{\nabla}\Phi = \partial_r \Phi(r)\,\hat{r} = \left[\frac{f'(r)}{r} - \frac{f(r)}{r^2}\right]\hat{r}$$

$$Q_R = -\varepsilon_0\left[\frac{f'(R)}{R} - \frac{f(R)}{R^2}\right]4\pi R^2 = -4\pi\varepsilon_0\left[f'(R)R - f(R)\right]$$

Let $R \to \infty$ to encompass the whole system (since $f(R) \to B$, $f'(R) \to 0$):

$$Q_{tot} = \lim_{R\to\infty} Q_R = 4\pi\varepsilon_0 B \quad \text{if } f'(R) \to 0 \text{ faster than } \frac{1}{R} \text{ as } R \to \infty$$

$$Q_{tot} = 4\pi\varepsilon_0 B$$

b) The potential is singular at the origin, so let's use Q_R again but take $R \to 0$:

$$q_{origin} = \lim_{R\to 0} Q_R = 4\pi\varepsilon_0 A \quad \text{if } f'(R) \text{ does not diverge or diverges slower than } \pm\frac{1}{R} \text{ as } R \to 0$$

$$q_{origin} = 4\pi\varepsilon_0 A$$

c) $\rho(\vec{r}) = \rho_d(r) + 4\pi\varepsilon_0 A\,\delta(\vec{r})$ where ρ_d is the non-singular distribution

$$\rho(\vec{r}) = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = -\varepsilon_0 \vec{\nabla}\cdot(\vec{\nabla}\Phi) = -\varepsilon_0 \vec{\nabla}\cdot\left[f'(r)\frac{\hat{r}}{r} - f(r)\frac{\hat{r}}{r^2}\right]$$

$$= -\varepsilon_0\left[\left(\vec{\nabla}f'(r)\cdot\frac{\hat{r}}{r} + f'(r)\,\vec{\nabla}\cdot\frac{\hat{r}}{r}\right) - \left(\vec{\nabla}f(r)\cdot\frac{\hat{r}}{r^2} + f(r)\,\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)\right]$$

(since $\vec{\nabla}\cdot(\psi\vec{a}) = \vec{\nabla}\psi\cdot\vec{a} + \psi\,\vec{\nabla}\cdot\vec{a}$)

$$= -\varepsilon_0\left[f''(r)\frac{1}{r} + \cancel{f'(r)\frac{1}{r^2}} - \cancel{f'(r)\frac{1}{r^2}} - \overset{A}{\cancel{f(r)}}\,4\pi\,\delta(\vec{r})\right]$$

$$= -\varepsilon_0\frac{f''(r)}{r} + 4\pi\varepsilon_0 A\,\delta(\vec{r})$$