10. *Electricity and Magnetism* (Fall 2006)

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

(a) What is the total charge of this system? Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

(b) Identify any point charges in this system and give their location and charge.

(c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

a) Letting $r \to \infty$, $\quad \Phi \to \frac{B}{r} \quad \Rightarrow \quad \boxed{Q_{tot} = \frac{B}{K}}$ where $K = \frac{1}{4\pi\varepsilon_0}$

b) Since the potential is singular at $r=0$, the origin is suspect:

Letting $r \to 0$, $\quad \Phi \to \frac{A}{r} \quad \Rightarrow \quad \boxed{q_{origin} = \frac{A}{K}}$

There are no other singularities in Φ, so there are no other point charges.

c) $\rho(\vec{r}) = \rho_d(r) + \frac{A}{K}\delta(\vec{r})$, where ρ_d is the non-singular distribution

$$\rho(\vec{r}) = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = -\varepsilon_0 \vec{\nabla}\cdot(\vec{\nabla}\Phi) = -\varepsilon_0 \vec{\nabla}\cdot\left(\partial_r \frac{f(r)}{r}\hat{r}\right)$$

$$= -\varepsilon_0 \vec{\nabla}\cdot\left(f'(r)\frac{\hat{r}}{r} - f(r)\frac{\hat{r}}{r^2}\right)$$

$$= -\varepsilon_0\left(\vec{\nabla}f'(r)\cdot\frac{\hat{r}}{r} + f'(r)\vec{\nabla}\cdot\frac{\hat{r}}{r} - \vec{\nabla}f(r)\cdot\frac{\hat{r}}{r^2} - f(r)\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)$$

$$= -\varepsilon_0\left(f''(r)\hat{r}\cdot\frac{\hat{r}}{r} + f'(r)\left[\frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\frac{1}{r}\right)\right] - f'(r)\hat{r}\cdot\frac{\hat{r}}{r^2} - f(r)\,4\pi\delta(\vec{r})\right)$$

$$= -\varepsilon_0\left(\frac{f''(r)}{r} + \cancel{\frac{f'(r)}{r^2}} - \cancel{\frac{f'(r)}{r^2}} - 4\pi f(0)\delta(\vec{r})\right)$$

$$= -\varepsilon_0\frac{f''(r)}{r} + 4\pi\varepsilon_0 A\,\delta(\vec{r})$$

$$= -\varepsilon_0\frac{f''(r)}{r} + \frac{A}{K}\delta(\vec{r})$$

Note: Care was needed in dealing with the divergence to retain the Dirac delta.