5. *Quantum Mechanics* (Fall 2006)

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r}$$

(a) Imagine you could find the solution $\psi(\mathbf{r})$ to the time-independent Schrödinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\mathbf{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

(b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

(c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

a) Scattering off heavy nucleus $\Rightarrow$ elastic scattering off fixed scatterer/potential $(k' = k)$

singular potential (at origin) $\Rightarrow$ strong potential

$\Rightarrow$ (I would think) partial wave analysis rather than the Born approximation, but perhaps the singularity is "small enough" to use the Born approximation

$$f(\theta,\phi) = -\frac{(2\pi)^{3/2}}{4\pi}\frac{2M}{\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}'}\, V(\vec{r}')\, \psi(\vec{r}')\, d^3r'$$

where $\vec{k}' = k\hat{r}$ is the scattering wave vector and $k = |\vec{k}| = |k\hat{z}|$ is the incident wave number

b) Born approx: $\psi(\vec{r}) \approx \psi_0(\vec{r}) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}} = (2\pi)^{-3/2} e^{ikz}$

$$f^{(1)}(\theta,\phi) = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}'}\, V(r')\, e^{i\vec{k}\cdot\vec{r}'}\, d^3r'$$

$$= -\frac{2M}{\hbar^2}\frac{1}{q}\int_0^\infty r' V(r') \sin(qr')\, dr'$$

(by spherical symmetry of potential... skipped steps)
$\vec{q} \equiv \vec{k} - \vec{k}'$ $\quad q = |\vec{k} - \vec{k}'| = 2k \sin\frac{\theta}{2}$

$$= -\frac{2M}{\hbar^2 q}\int_0^\infty \frac{V_0}{\mu} e^{-\mu r'} \sin(qr')\, dr'$$

$$= -\frac{2MV_0}{\hbar^2 \mu q}\int_0^\infty \left[qr' - \frac{1}{3!}q^3 r'^3 + \frac{1}{5!}q^5 r'^5 - \dots\right] e^{-\mu r'}\, dr'$$

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b) (continued)

$$= -\frac{2MV_0}{\hbar^2 \mu q}\left[q\left(\frac{1}{\mu}\right)^2 - \frac{1}{3!} q^3 \left(\frac{1}{\mu}\right)^4 3! + \frac{1}{5!} q^5 \left(\frac{1}{\mu}\right)^6 5! - \cdots \right]$$

since $\int_0^\infty x^n e^{-x/a} dx = a^{n+1}\Gamma(n+1) = a^{n+1} n!$

$$= -\frac{2MV_0}{\hbar^2 \mu q}\frac{1}{q}\left[\left(\frac{q}{\mu}\right)^2 - \left(\frac{q}{\mu}\right)^4 + \left(\frac{q}{\mu}\right)^6 - \cdots \right] = +\frac{2MV_0}{\hbar^2 \mu q^2}\sum_{n=1}^{\infty}\left[-\left(\frac{q}{\mu}\right)^2\right]^n$$

$$= \frac{2MV_0}{\hbar^2 \mu q^2}\frac{-\left(\frac{q}{\mu}\right)^2}{1-\left[-\left(\frac{q}{\mu}\right)^2\right]}$$ assuming $\frac{q}{\mu} < 1 \Rightarrow 2k \sin\frac{\theta}{2} < \mu \Rightarrow k < \frac{\mu}{2}$

$$= -\frac{2MV_0}{\hbar^2 \mu q^2}\frac{\left(\frac{q}{\mu}\right)^2}{1+\left(\frac{q}{\mu}\right)^2} = -\frac{2MV_0}{\hbar^2 \mu q^2}\frac{q^2}{\mu^2+q^2} = -\frac{2MV_0}{\hbar^2 \mu}\frac{1}{\mu^2+q^2}$$

$$\Rightarrow f^{(1)}(\theta,\phi) = f^{(1)}(\theta) = -\frac{2MV_0}{\hbar^2 \mu}\frac{1}{\mu^2 + 4k^2\sin^2\frac{\theta}{2}}$$

c) $E \to 0$ $\quad E = \frac{p^2}{2M} = \frac{\hbar^2 k^2}{2M} \Rightarrow k \to 0 \quad (\lambda \to \infty)$

$$\Rightarrow f^{(1)}(\theta) \to -\frac{2MV_0}{\hbar^2\mu^3} = -f_0$$

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta,\phi)|^2 d\Omega \to \int |f_0|^2 d\Omega = 4\pi |f_0|^2$$

$$\sigma_{tot} \to 4\pi\, 4\left(\frac{MV_0}{\hbar^2\mu^3}\right)^2 = \pi\left(\frac{4MV_0}{\hbar^2\mu^3}\right)^2$$