4. *Quantum Mechanics* (Fall 2006)

Consider a quantum mechanical system with Hamiltonian

$$H = a^\dagger a$$

Where a and $a^\dagger$ are operators satisfying the following relations

$$a^2 = 0, \quad (a^\dagger)^2 = 0 \quad a^\dagger a + a a^\dagger = 1$$

(a) Show that the Hamiltonian satisfies

$$H^2 = H$$

(b) Find the eigenvalues of the Hamiltonian H.

(c) If $|0\rangle$ is the **unique** normalized ground state of the system (i.e., the state with the lowest energy eigenvalue) find

$$a|0\rangle = \ ?$$

Under the assumption above, what dimension can the complete Hilbert space of states have?

a) $H^2 = (a^+a)^2 = a^+a\,a^+a = a^+(1-a^+a)a = a^+a - \cancel{a^{+2}}\,\cancel{a^2} = a^+a = H$

b) $H|E\rangle = E|E\rangle \Rightarrow H^2|E\rangle = E^2|E\rangle$

$= H|E\rangle = E|E\rangle$

$\Rightarrow E^2 = E \Rightarrow E \in \{0, 1\}$

c) $H|0\rangle = E_0|0\rangle = 0|0\rangle = 0 \qquad H|1\rangle = E_1|1\rangle = |1\rangle$

$H(a|0\rangle) = a^+aa|0\rangle = a^+\cancel{a^2}|0\rangle = 0 \Rightarrow$ either $a|0\rangle = 0$ or $a|0\rangle \propto |0\rangle$

$Ha = a^+aa = a^+\cancel{a^2} = 0 = q(0) = qHa$ with q arbitrary

$[H,a] = Ha - aH = \cancel{a^+a^2} - aa^+a = -a(1-aa^+) = -a + \cancel{a^2}a^+ = -a$

$H(a|0\rangle) = (Ha)|0\rangle = q(Ha)|0\rangle = q([H,a] + aH)|0\rangle = q(-a + a\cancel{H})|0\rangle$

$= -q(a|0\rangle) = 0$

with q arbitrary

pick $q \neq 0 \Rightarrow a|0\rangle = 0$

The complete Hilbert space of states could be 2D (if the E=1 state is nondegenerate) or (n+1)D (if the E=1 state is degenerate with multiplicity n).