3. *Quantum Mechanics* (Fall 2006)

Consider two flavours of massive neutrinos, denote $|\nu_e\rangle$ the electron neutrino flavour eigenstateand $|\nu_\mu\rangle$ the muon neutrino flavour eigenstate. These are related to the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ by

$$\begin{aligned} |\nu_e\rangle &= \cos(\theta)\,|\nu_1\rangle - \sin(\theta)\,|\nu_2\rangle \\ |\nu_\mu\rangle &= \sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle \end{aligned}$$

(a) Show that flavour eigenstates and energy eigenstates are related by a unitary transformation.

(b) The energy of the eigenstate $|\nu_i\rangle$ is

$$E_i = \sqrt{\mathbf{p}^2c^2 + m_i{}^2c^4}, \quad i = 1, 2$$

Assume that an electron neutrino is produced in the sun with momentum $\mathbf{p}$ such that $|\mathbf{p}| \gg m_i c$. Find the probability for the electron neutrino to oscillate into a muon neutrino after travelling a distance L.

a) $\vec{\Psi}_f = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{\Psi}_e$ $\qquad |\Psi\rangle = \vec{\Psi}_f \cdot |\vec{\beta}_f\rangle = \Psi_{f1}|\nu_e\rangle + \Psi_{f2}|\nu_\mu\rangle$

$= U\vec{\Psi}_e$ $\qquad = \vec{\Psi}_e \cdot |\vec{\beta}_e\rangle = \Psi_{e1}|\nu_1\rangle + \Psi_{e2}|\nu_2\rangle$

where, e.g., $\vec{\Psi}_f = \begin{pmatrix} \Psi_{f1} \\ \Psi_{f2} \end{pmatrix}$ $\quad |\vec{\beta}_f\rangle = \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$

$$U^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow U^\dagger U = \begin{pmatrix} C^2+S^2 & -SC+CS \\ -CS+SC & -(-S^2)+C^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$\Rightarrow U^\dagger = U^{-1} \Rightarrow U$ is unitary

b) $|\Psi_0\rangle = |\nu_e\rangle$ with $|\vec{p}| >> m_1 c, m_2 c$, traveling $L \Rightarrow \Delta t \approx \frac{L}{c} \equiv \tau$

$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi_0\rangle = e^{-i\omega_1 t}\cos\theta|\nu_1\rangle - e^{-i\omega_2 t}\sin\theta|\nu_2\rangle$ where $\omega_i \equiv \frac{E_i}{\hbar}$

$$P_{\nu_e\to\nu_\mu}(L) = |\langle\nu_\mu|\Psi(\Delta t)\rangle|^2$$

$$\langle\nu_\mu|\Psi(t)\rangle = (\sin\theta \;\; \cos\theta)\begin{pmatrix} e^{-i\omega_1 t}\cos\theta \\ -e^{-i\omega_2 t}\sin\theta \end{pmatrix} = (e^{-i\omega_1 t} - e^{-i\omega_2 t})\sin\theta\cos\theta$$

$$= \tfrac{1}{2}\sin 2\theta\,(e^{-i\omega_1 t} - e^{-i\omega_2 t})$$

$$P_{\nu_e\to\nu_\mu}(L) = \tfrac{1}{4}\sin^2 2\theta\left[1 - e^{i(\omega_2-\omega_1)\tau} - e^{-i(\omega_2-\omega_1)\tau} + 1\right] = \tfrac{1}{4}\sin^2 2\theta\left[2 - 2\cos(\omega_2-\omega_1)\tau\right]$$

$$= \sin^2 2\theta\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(\omega_2-\omega_1)\tau\right) = \sin^2 2\theta\,\sin^2\left(\tfrac{1}{2}(\omega_2-\omega_1)\tau\right)$$

$$\left(\begin{aligned} E_2 - E_1 &= \sqrt{p^2c^2+m_2^2c^4} - \sqrt{p^2c^2+m_1^2c^4} = pc\left[(1+m_2^2c^2/p^2)^{1/2} - (1+m_1^2c^2/p^2)^{1/2}\right] \quad \frac{m_i c}{p} << 1 \\ &\approx pc\left[1 + \tfrac{1}{2}m_2^2c^2/p^2 - 1 - \tfrac{1}{2}m_1^2c^2/p^2\right] = c^3\Delta(m^2)/2p \end{aligned}\right)$$

$$\approx \sin^2 2\theta\,\sin^2\left[c^2\Delta(m^2)L/4\hbar p\right]$$