2. *Quantum Mechanics* (Fall 2006)

The spin degree of freedom of a spin 1/2 particle with mass m can be described in a basis $|\pm\rangle$, where

$$\sigma_3 |+\rangle = +|+\rangle, \quad \sigma_3 |-\rangle = -|-\rangle,$$

and where σ_3 is the third Pauli matrix. The spin operator for a single fermion is $S_3 = \frac{\hbar}{2}\sigma_3$.

(a) Two identical fermions of spin 1/2 are initially assumed to be noninteracting. For this part of the problem take only the spin degrees of freedom into account. Construct the singlet state, i.e., the state for which the total spin of the two fermion system satisfies $S_3 = 0$ and $S^2 = 0$.

Now consider that the two spin 1/2 fermions are both moving in a one dimensional infinite square well with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ -a & 0 < x < L \\ \infty & x > L \end{cases}$$

For the rest of the problem take both the spin degrees of freedom and the spatial wavefunction into account.

(b) What does the Fermi exclusion principle imply for the wavefunction of the two fermion system? What does this imply for the spatial wavefunctions of the singlet state?

(c) Find the normalized wavefunction of the two fermion system which has the lowest energy and is a singlet. Find the energy eigenvalue for this state.

(d) Now assume that there is a small interaction of the form

$$V_{\text{int}}(x_1, x_2) = -\alpha\, \delta(x_1 - x_2)$$

To lowest order in perturbation theory find the change in energy of the ground state due to the interaction.

a) $|m_1, m_2\rangle \to |j, m\rangle' \qquad m = m_1 + m_2 \qquad -j \leq m \leq j$

$|0,0\rangle' = ?$

$|1,1\rangle' = |++\rangle$ by necessity

$|1,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$ using $J_- = J_{1-} + J_{2-}$ $*$

$\boxed{|0,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)}$ by orthogonality to $|1,0\rangle'$ $\circledast$

The same can be found using $|1,-1\rangle' = |--\rangle$ and $J_+ = J_{1+} + J_{2+}$

$*$ $J_-|1,1\rangle' = \sqrt{1(1+1) - 1(1-1)}\,\hbar\, |1,0\rangle' = \sqrt{2}\,\hbar\, |1,0\rangle'$

$= (J_{1-} + J_{2-})|++\rangle = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\,\hbar\,(|-+\rangle + |+-\rangle)$

$= \hbar(|+-\rangle + |-+\rangle)$

$\Rightarrow |1,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

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a) (continued)

⊗ and noting that $|1,0\rangle'$ must incorporate both indistinguishable states $|+-\rangle$ and $|-+\rangle$

b) 2 identical spin-1/2 particles

$\Rightarrow$ fermionic odd (antisymmetric) exchange parity

$$P_x\,\Upsilon(x_1,x_2,m_1,m_2,t) = \boxed{\Upsilon(x_2,x_1,m_2,m_1,t) = -\Upsilon(x_1,x_2,m_1,m_2,t)}$$

$\Rightarrow$ Pauli exclusion principle ("Fermi exclusion principle"?)

$$H \neq H(\text{spin}) \Rightarrow \Upsilon(x_1,x_2,m_1,m_2,t) = \Psi(x_1,x_2,t)\,\chi(m_1,m_2) \quad \text{(separable)}$$

Singlet state $\chi(+,-) = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ (more properly $\chi(\{m_1,m_2\})$)

is odd under P_x and Υ is odd under P_x,

so Ψ must be even under P_x : $\boxed{\Psi(x_2,x_1,t) = +\Psi(x_1,x_2,t)}$

c) $H \neq H(\text{interaction}, t) \Rightarrow \Psi(x_1,x_2,t) = \psi_1(x_1)\,\psi_2(x_2)\,\tau(t)$

$H = H_1 + H_2 \qquad H_i = \vec{p}_i^{\,2}/2m + V(x_i) \qquad \tau = e^{-iEt/\hbar}$

$$H_i\psi_i = \left[-\frac{\hbar^2}{2m}d_{x_i}^{\,2} - a\right]\psi_i = E_i\psi_i \quad \text{for } 0 \le x_i \le L, \quad E = E_1 + E_2$$

$$\Rightarrow \left[d_{x_i}^{\,2} + \frac{2m}{\hbar^2}(E_i + a)\right]\psi_i = 0 \quad \Rightarrow \quad \psi_i(x_i) = A_i \sin(k_i x_i) \text{ due to B.C.s}$$

where $k_i^{\,2} = \frac{2m}{\hbar^2}(E_i + a) = \left(\frac{n_i\pi}{L}\right)^2$

and $n_i \in \mathbb{Z}^+$

$$\int_0^L\!\!\int_0^L |\Psi(x_1,x_2,t)|^2\,dx_1\,dx_2 = 1 \quad \Rightarrow \quad \int_0^L |\psi_i(x_i)|^2\,dx_i = 1 \Rightarrow A_i = \sqrt{\tfrac{2}{L}}$$

$$\Rightarrow \Upsilon_{n_1 n_2}(x_1,x_2,m_1,m_2,t) = \left(\tfrac{2}{L}\right)\sin\left(\tfrac{n_1\pi}{L}x_1\right)\sin\left(\tfrac{n_2\pi}{L}x_2\right)e^{-iEt/\hbar}\,\chi(m_1,m_2)$$

$$E = E_{n_1 n_2} = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2(n_1^{\,2} + n_2^{\,2}) - 2a$$

(in full detail, you'll find $E_1 = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2 n_1^{\,2} - a - U$ and $E_2 = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2 n_2^{\,2} - a + U$, where U is arbitrary)

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c) (continued)

Lowest singlet energy eigenstate

$\Rightarrow$ spatially P_x-even state with lowest $E_{n_1 n_2}$

$\Rightarrow$ $n_1 = n_2$ and $n_1 = n_2 = 1$

$$\Rightarrow \quad \Upsilon_{11}(x_1, x_2, +, -, t) = \left(\tfrac{2}{L}\right) \sin\left(\tfrac{\pi}{L}x_1\right) \sin\left(\tfrac{\pi}{L}x_2\right) e^{-iE_{11}t/\hbar} \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

with $E_{11} = \frac{1}{m}\left(\frac{\hbar\pi}{L}\right)^2 - 2a$

d) $$\Delta E^{(1)} = \langle \Upsilon | V_{int} | \Upsilon \rangle = -\alpha \int_0^L \int_0^L \left(\tfrac{2}{L}\right)^2 \sin^2\left(\tfrac{\pi}{L}x_1\right) \sin^2\left(\tfrac{\pi}{L}x_2\right) \delta(x_1 - x_2)\, dx_1\, dx_2$$

$$= -\frac{4\alpha}{L^2} \int_0^L \sin^4\left(\tfrac{\pi}{L}x\right) dx \quad \text{where} \quad \sin^4\theta = \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)^2 = \tfrac{1}{4} - \cos 2\theta + \tfrac{1}{4}\cos^2 2\theta$$

$$= -\frac{4\alpha}{L^2} \int_0^L \left[\tfrac{1}{4} - \cos\left(\tfrac{2\pi}{L}x\right) + \tfrac{1}{4}\cos^2\left(\tfrac{2\pi}{L}x\right)\right] dx$$

$$= -\frac{4\alpha}{L^2}\left[\frac{L}{4} - \underbrace{\cancel{\left[\frac{L}{2\pi}\sin\left(\tfrac{2\pi}{L}x\right)\right]_0^L}}_{0} + \frac{1}{4}\left(\frac{L}{2}\right)\right] = -\frac{4\alpha}{L^2}\frac{L}{8} = -\frac{\alpha}{2L}$$