1. *Quantum Mechanics* (Fall 2006)

(a) For a spherically symmetric potential, show that the radial part of a wave function obeys the radial Schrödinger equation

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{2mr^2}+V(r)-E\right)u_l(r)=0$$

Assume that the potential $V(r)$ vanishes rapidly for large r and is less singular than $1/r^2$ for small r.

(b) Derive the behavior of $u_l(r)$ for $r \to 0$.

(c) Derive the behavior of $u_l(r)$ for large r ($r \to \infty$) when it describes a bound state.

a) $\hat{H}\Psi = \hat{E}\Psi \qquad \Psi = \Psi(\vec{r}, t)$

$$\left[\frac{\hat{p}^2}{2m} + \hat{V}\right]\Psi = \hat{E}\Psi$$

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\Psi = i\hbar\,\partial_t\,\Psi$$

Assume $\Psi(\vec{r},t) = \psi(\vec{r})\,\tau(t) = R(r)\,Y(\theta,\phi)\,\tau(t)$

where a general solution is a linear combination (add, integrate) of terms of this type

$$\frac{\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\psi(\vec{r})}{\psi(\vec{r})} = \frac{i\hbar\,d_t\,\tau(t)}{\tau(t)} \equiv E$$

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\vec{\nabla}^2 = \left[\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right\}\right]$$

$$\equiv \left[\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{L^2(\theta,\phi)}{r^2}\right]$$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\left\{\partial_r(r^2\partial_r) - L^2(\theta,\phi)\right\} + V(r)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\frac{\left[-\frac{\hbar^2}{2m}d_r(r^2 d_r) + r^2V(r) - r^2E\right]R(r)}{R(r)} = -\frac{\frac{\hbar^2}{2m}L^2(\theta,\phi)\,Y(\theta,\phi)}{Y(\theta,\phi)} = -\frac{\hbar^2\,l(l+1)}{2m},$$

using the lemma that $L^2(\theta,\phi)\,Y(\theta,\phi) = l(l+1)\,Y(\theta,\phi)$, $l \in \mathbb{N} = \{0,1,2,\ldots\}$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}d_r(r^2d_r) + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right]R_l(r) = 0$$

multiplying by $r \Rightarrow \left[-\frac{\hbar^2}{2m}\frac{1}{r}d_r(r^2d_r) + \left\{\frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right\}r\right]R_l(r) = 0$

and defining $u_l(r) = r\,R_l(r)$ so that

$d_r^2 u_l = d_r^2(rR_l) = d_r(R_l + r\,d_rR_l) = 2\,d_rR_l + r\,d_r^2R_l$ and

$\frac{1}{r}d_r(r^2d_r)R_l = \frac{1}{r}(2r\,d_rR_l + r^2d_r^2R_l) = 2\,d_rR_l + r\,d_r^2R_l,$ we have

$$\left[-\frac{\hbar^2}{2m}d_r^2 + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right]u_l(r) = 0 \qquad ✓$$

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b) For $r \to 0$, the equation is dominated by the $\frac{\hbar^2 \ell(\ell+1)}{r^2}$ term (since $V(r)$ is less singular than $1/r^2$ for small r):

$$\frac{\hbar^2}{2m}\left[-d_r^2 + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = 0$$

Assuming $u_\ell(r) = \sum_{k=1}^{\infty} a_k r^k$ (there can be no nonpositive powers since $u_\ell(0) = 0 R_\ell(0) = 0$), we have

$$\left[-\sum_{k=}^{\infty} k(k-1) a_k r^{k-2} + \ell(\ell+1) \sum_{k=}^{\infty} a_k r^{k-2}\right] = 0$$

$\Rightarrow \quad k(k-1) = \ell(\ell+1) \quad \Rightarrow \quad k = \ell+1, -\ell$

but $-\ell$ is not possible since $\ell \in \mathbb{N}$ and $k \geq 1$

$\Rightarrow \quad u_\ell(r) \propto r^{\ell+1}$ asymptotically as $r \to 0$

c) For $r \to \infty$ and $E < 0$ (bound states), the $\frac{\hbar^2 \ell(\ell+1)}{r^2}$ term and $V(r)$ vanish (since $V(r)$ vanishes rapidly for large r):

$$\left[-\frac{\hbar^2}{2m} d_r^2 - E\right] u_\ell(r) = 0$$

$\Rightarrow \quad d_r^2 u_\ell(r) = -\frac{2mE}{\hbar^2} u_\ell(r) \equiv k^2 u_\ell(r)$ for $k = \sqrt{-\frac{2mE}{\hbar^2}} \in \mathbb{R}$

$\Rightarrow \quad u_\ell(r) \propto e^{kr}$ asymptotically as $r \to \infty$