14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature T. What is the average energy per photon in units of kT?

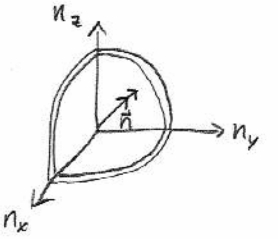
You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$

Consider a cubical cavity for simpliety. The photons oscillate in distinct modes:

$$\epsilon = pc = \hbar k c = \hbar c \sqrt{(n_x \pi/L)^2 + (n_y \pi/L)^2 + (n_z \pi/L)^2} = \frac{\pi \hbar c}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\pi \hbar c}{L} n$$

$$\langle \epsilon \rangle = \sum_i \epsilon_i P(\epsilon_i) = \frac{\int_0^\infty \epsilon\, f(\epsilon)\, w(\epsilon)\, d\epsilon}{\int_0^\infty f(\epsilon)\, w(\epsilon)\, d\epsilon} \qquad f(\epsilon) = \frac{1}{e^{-\beta\epsilon} - 1} \text{ for photons}$$



$$\#\text{ states} = \rho(n)\, dn = \frac{1}{8} 4\pi n^2 dn = w(\epsilon)\, d\epsilon$$

$$n = \frac{L}{\pi \hbar c}\epsilon \qquad dn = \frac{L}{\pi \hbar c} d\epsilon$$

$$\Rightarrow w(\epsilon)\, d\epsilon = \frac{1}{8} 4\pi \left(\frac{L}{\pi \hbar c}\right)^3 \epsilon^2 d\epsilon = \frac{V}{2\pi^2} \frac{\epsilon^2}{(\hbar c)^3} d\epsilon$$

$$I_1 \equiv \int_0^\infty \epsilon\, f(\epsilon)\, w(\epsilon)\, d\epsilon = \frac{V}{2\pi^2 (\hbar c)^3}\left[\int_0^\infty \frac{\epsilon^3 d\epsilon}{e^{-\beta\epsilon} - 1} = \frac{1}{\beta^4}\int_0^\infty \frac{x^3 dx}{e^{-x} - 1} \approx 6.5 (kT)^4\right] \qquad x \equiv \beta\epsilon$$

$$I_2 \equiv \int_0^\infty f(\epsilon)\, w(\epsilon)\, d\epsilon = \frac{V}{2\pi^2 (\hbar c)^3}\left[\int_0^\infty \frac{\epsilon^2 d\epsilon}{e^{-\beta\epsilon} - 1} = \frac{1}{\beta^3}\int_0^\infty \frac{x^2 dx}{e^{-x} - 1} \approx 2.4 (kT)^3\right]$$

$$\Rightarrow \langle \epsilon \rangle = \frac{I_1}{I_2} \approx \frac{6.5 (kT)^4}{2.4 (kT)^3} \approx 2.7 (kT)$$