14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature T. What is the average energy per photon in units of kT?

You may find the following formulae useful:

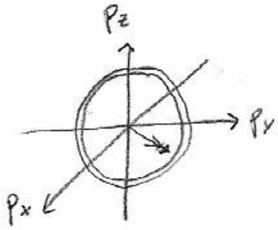
$$\int_0^\infty \frac{x^3dx}{e^x-1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2dx}{e^x-1} \approx 2.4$$

$$\epsilon = pc$$

$$\langle \epsilon \rangle = \sum_i \epsilon_i P(\epsilon_i) = \frac{\int \epsilon f(\epsilon) w(\epsilon) d\epsilon}{\int f(\epsilon) w(\epsilon) d\epsilon}$$

for photons $f(\epsilon) = \frac{1}{e^{\beta\epsilon}-1}$

$$w(\epsilon) = \frac{d\Omega}{d\epsilon} = \frac{\frac{1}{h^3} dV_{phase}}{d\epsilon}$$



$$dV_{phase} = \int_{\mathbb{R}^3} d^3r \int_{S_{octant}} p^2 d\Omega_p \, dp = (V)(4\pi p^2)dp$$

$$= \frac{4\pi V}{c^3}\epsilon^2 d\epsilon \quad \Rightarrow \quad w(\epsilon) = \frac{4\pi V}{(hc)^3}\epsilon^2$$

$$\langle \epsilon \rangle = \frac{\frac{4\pi V}{(hc)^3}\int \frac{\epsilon^3 d\epsilon}{e^{-\beta\epsilon}-1}}{\frac{4\pi V}{(hc)^3}\int \frac{\epsilon^2 d\epsilon}{e^{-\beta\epsilon}-1}} = \frac{\left(\frac{1}{\beta}\right)^4 \int \frac{x^3 dx}{e^{-x}-1}}{\left(\frac{1}{\beta}\right)^3 \int \frac{x^2 dx}{e^{-x}-1}}$$

with $x \equiv \beta\epsilon$, $\epsilon = \frac{x}{\beta}$

$$\approx \left(\frac{6.5}{2.4}\right)\left(\frac{1}{\beta}\right) \approx 2.7\,(kT)$$