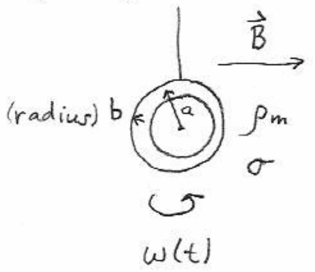
11. *Electricity and Magnetism* (Fall 2005)

A thin copper circular ring (conductivity σ, *mass* density ρ_m) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. The initial rotation frequency is ω_0. Calculate the time it takes for the frequency to decrease to $1/e$ of its original value, assuming the energy all goes into Joule heating. (Assume the requested time τ is large compared to the rotation period.)



$$\omega(t=0) = \omega_0 \qquad \omega(\tau) = \frac{1}{e}\omega_0 \qquad \tau = ?$$

$$E(t) = \frac{1}{2} I\,\omega^2(t) \qquad (1)$$

$$I = \int r^2\,dm \approx a^2 M = a^2 \rho_m V = a^2 \rho_m (\pi b^2 2\pi a)$$
$$= 2\pi^2 \rho_m b^2 a^3$$

Joule heating (averaged over many cycles):

$$\frac{dE}{dt} = -\frac{\overline{V^2(t)}}{R} \qquad R = \rho\frac{L}{A} \approx \frac{1}{\sigma}\frac{2\pi a}{\pi b^2} = \frac{2a}{\sigma b^2}$$

$$V(t) = -\partial_t \Phi_B(t) = -\partial_t\left(\vec{B}\cdot\vec{A}\right) = -\partial_t\left(B\pi a^2 \sin[\omega(t)\,t]\right)$$
$$= -B\pi a^2 \cos[\omega(t)\,t]\,\frac{d\omega}{dt}$$

$$\frac{dE}{dt} = -\frac{(B\pi a^2)^2}{R}\,\overline{\cos^2[\omega(t)\,t]\left(\frac{d\omega}{dt}\right)^2} = -\frac{(B\pi a^2)^2}{2R}\left(\frac{d\omega}{dt}\right)^2 \equiv -K\dot{\omega}^2$$

Using (1) above:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = I\omega\dot{\omega} = -K\dot{\omega}^2$$

$$\Rightarrow\ -\frac{K}{I}\dot{\omega} = \omega \qquad \Rightarrow \qquad \omega = \omega_0 e^{-\frac{K}{I}t}$$

$$\Rightarrow\ \tau = \frac{I}{K} = \frac{(2\cancel{\pi^2}\rho_m \cancel{b^2} a^3)}{B^2 \cancel{\pi^2} a^4}\,2\left(\frac{2a}{\sigma \cancel{b^2}}\right) = \frac{8\rho_m}{\sigma B^2}$$