10. *Electricity and Magnetism* (Fall 2005)

A pulsar emits bursts of radio waves, which are observed from Earth at two different frequencies, say ω_1 and ω_2. An astronomer notes that the arrival time of the bursts is delayed at the lower frequency: the pulse at ω_1 arrives after the pulse at ω_2. The delay, τ, is due to dispersion in the interstellar medium. Assuming this medium consists of ionized hydrogen, estimate the distance s of the pulsar from the earth, as follows:

(a) Show that the electron plasma frequency for the dilute plasma — consisting of (heavy) ions and free electrons — is

$$\omega_p = \left(\frac{4\pi N e^2}{m_e}\right)^{1/2}$$

in e.s.u. Here N is the number of electrons per unit volume.

(b) Show that the index of refraction of the plasma is

$$n = \sqrt{\epsilon} = \left(1 - \frac{{\omega_p}^2}{\omega^2}\right)^{1/2}$$

Hint: Write the equation of motion for a free electron in an oscillating ($e^{-i\omega t}$) electric field and find the plasma's polarizability χ. Then $\epsilon = 1 + 4\pi\chi$.

(c) From the relation above find the group velocity of the light, and use this result to find the distance to the pulsar. (You may assume the frequencies are large compared to ω_p.)

a) The plasma frequency is the natural frequency* of charge density oscillations in a conductor (e.g. plasma, metal). If we assume that ρ is oscillating with no externally applied field and ignore damping and any restoring force, we may easily derive ω_p.

$\rho(\vec{x},t) = \rho_0(\vec{x})\, e^{-i\omega_p t}$ where $\rho_0(\vec{x})$ varies positively and negatively due to the displacement of electrons against a constant proton charge density, but over large scales averages to zero. ($\rho_0(\vec{x})$ must vary in sign for charge to be conserved.)

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{E}(\vec{x},t) = \vec{E}_0(\vec{x})\, e^{-i\omega_p t}$

Take $\vec{d}$ to be the displacement an electron near position $\vec{x}$:

$m_e \ddot{\vec{d}} = \vec{F}_{net} = -e\vec{E}(\vec{x},t) = -e\vec{E}_0(\vec{x})\, e^{-i\omega_p t}$

The velocity of that electron is thus $\vec{v} = \dot{\vec{d}} = \cancel{\vec{v}_0} + \frac{e}{i\omega_p m_e}\vec{E}_0(\vec{x})\, e^{-i\omega_p t}$ $\vec{0}$

The current is due to the motion of the electrons, since the protons are essentially stationary, and the electrons have an average number density of $n_e = N$ (so an average charge density $\bar{\rho}_e = -n_e e$).

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a) (continued)

$$\Rightarrow \quad \underset{\sim}{J}(\vec{x}) \approx \bar{\rho}_e \, \underset{\sim}{U}(\vec{x}) = (-n_e e)\left[\frac{e}{i\omega_p m_e}\vec{E}_0(\vec{x})\, e^{-i\omega_p t}\right]$$

Finally, we use continuity:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \quad \Rightarrow \quad -\frac{n_e e^2}{i\omega_p m_e}\cancelto{\rho_0(\vec{x})/\varepsilon_0}{\vec{\nabla}\cdot\vec{E}_0(\vec{x})}\, \cancel{e^{-i\omega_p t}} = +i\omega_p \cancel{\rho_0(\vec{x})\, e^{-i\omega_p t}}$$

$$\Rightarrow \quad i\frac{n_e e^2}{\varepsilon_0 \omega_p m_e} = i\omega_p \quad \Rightarrow \quad \omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

In e.s.u. $\varepsilon_0 = \frac{1}{4\pi}$, and using $n_e = N$, $\quad \omega_p = \sqrt{\frac{4\pi N e^2}{m_e}}$ ✓

b) Now we include damping, restoring, and driving forces:

$$\underset{\sim}{P} = \varepsilon_0 \chi \underset{\sim}{E} \approx (-n_e e)\underset{\sim}{d} \qquad \underset{\sim}{E}(\vec{x},t) = \vec{E}_0(\vec{x})\, e^{-i\omega t} \qquad \left(\underset{\sim}{d} = -\frac{\varepsilon_0 \chi}{n_e e}\underset{\sim}{E}\right)$$

$$m_e \ddot{\underset{\sim}{d}} = \underset{\sim}{F}_{net} = -e\,\underset{\sim}{E}(\vec{x},t) - k_d \dot{\underset{\sim}{d}} - k_r \underset{\sim}{d}$$

$$\Rightarrow \quad m_e\left[\ddot{\underset{\sim}{d}} + \gamma_d \dot{\underset{\sim}{d}} + \omega_p^2 \underset{\sim}{d}\right] = -e\,\underset{\sim}{E} \qquad \text{and let } \underset{\sim}{d} = \underset{\sim}{d}_0\, e^{-i\omega t}$$

$$\left(\gamma_d \equiv \frac{k_d}{m_e} \qquad \omega_p^2 = \frac{k_r}{m_e}\right)$$

$$\Rightarrow \quad \left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]\underset{\sim}{d} = -\frac{e}{m_e}\underset{\sim}{E}$$

$$\Rightarrow \quad -\frac{\varepsilon_0 \chi}{n_e e} = -\frac{e}{m_e\left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]} \quad \Rightarrow \quad \chi = \frac{\cancelto{\omega_p^2}{\left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)}}{\left[-\omega^2 - i\omega\gamma_d + \omega_p^2\right]}$$

For $\omega \gg \gamma_d, \omega_p \qquad \chi = -\frac{\omega_p^2}{\omega^2} \qquad$ and $\qquad \varepsilon = \varepsilon_0(1+\chi) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

So $\quad n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$ ✓

c) $\quad v_g = \frac{d\omega}{dk}, \quad k^2 = \omega^2\mu\varepsilon \quad \Rightarrow \quad k = \omega\sqrt{\mu_0\varepsilon_0}\left(1 - \omega_p^2/\omega^2\right)^{1/2} = \frac{\omega}{c}\left(1 - \omega_p^2/\omega^2\right)^{1/2}$

$$= \frac{1}{c}\left(\omega^2 - \omega_p^2\right)^{1/2}$$

$$\Rightarrow \quad \omega = \left(c^2k^2 + \omega_p^2\right)^{1/2}$$

$$v_g = \frac{1}{\cancel{2}}\left(c^2k^2 + \omega_p^2\right)^{-1/2}\cancel{2}c^2k = \left((\omega^2 \cancel{-\omega_p^2}) \cancel{+\omega_p^2}\right)^{-1/2} c\left(\omega^2 - \omega_p^2\right)^{1/2}$$

$$= c\left(1 - \omega_p^2/\omega^2\right)^{1/2}$$

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c) (continued)

distance to pulsar:

$$D = v_{g1} t_1 = v_{g2} t_2 \qquad \tau = t_1 - t_2 \qquad t_1 > t_2 \qquad \omega_1 < \omega_2$$

$$v_{g1} < v_{g2}$$

$$D = v_{g1} t_1$$

$$D = v_{g2}(t_1 - \tau)$$

$$\begin{aligned} v_{g2} D &= \cancel{v_{g1} v_{g2}}\, t_1 \\ - v_{g1} D &= - \cancel{v_{g1} v_{g2}}\, t_1 + v_{g1} v_{g2} \tau \\ \hline (v_{g2} - v_{g1}) D &= v_{g1} v_{g2} \tau \end{aligned}$$

$$D = \frac{v_{g1} v_{g2} \tau}{(v_{g2} - v_{g1})} = \frac{\tau}{\left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}}\right)} = \frac{c\tau}{\left[(1 - \omega_p^2/\omega_1^2)^{-1/2} - (1 - \omega_p^2/\omega_2^2)^{-1/2}\right]}$$

$$\approx \frac{c\tau}{\left[(1 + \frac{1}{2}\omega_p^2/\omega_1^2) - (1 + \omega_p^2/\omega_2^2)\right]} \qquad \text{for } \omega_1, \omega_2 \gg \omega_p$$

$$= \frac{2c\tau}{\omega_p^2\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)}$$

* This definition of the plasma frequency corresponds to Jackson's assertions (pg 313, 3rd Ed.).