4. *Quantum Mechanics* (Fall 2005)

The Hamiltonian for a system consisting of three distinguishable spin half particles is

$$H = A(\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_3 \cdot \mathbf{s}_1)$$

where $\mathbf{s}_i$ is the spin of the i^{th} particle, and all the components of the spin of one particle commute with all the components of the spins of the others. What are the eigenvalues of H, and what are the degeneracies of each energy level?

It is easiest (I think) to analyze this Hamiltonian in terms of squares of spin operators. Since the operators commute with each other, we have $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + \vec{S}_3^{\,2} + 2(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1)$

so $H = \frac{A}{2}(\vec{S}^2 - \vec{S}_1^{\,2} - \vec{S}_2^{\,2} - \vec{S}_3^{\,2}) = \frac{A\hbar^2}{2}\left(s(s+1) - 3s_i(s_i+1)\right)$ where $s_i = \frac{1}{2}$

$= \frac{A\hbar^2}{2}\left(s(s+1) - \frac{9}{4}\right)$

Define $\vec{S}_{12} \equiv \vec{S}_1 + \vec{S}_2 \qquad m_{12} \equiv m_1 + m_2$

$\vec{S} = \vec{S}_{12} + \vec{S}_3 \qquad m = m_{12} + m_3$

s_i	m_i
$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$

$|s_1 - s_2| \le s_{12} \le s_1 + s_2$

$0 \le s_{12} \le 1$

$\Rightarrow s_{12} \in \{0, 1\}$

s_{12}	m_{12}
0	0
1	$-1, 0, 1$

$|s_{12} - s_3| \le s \le s_{12} + s_3$

1. $(s_{12} = 0)$ $\frac{1}{2} \le s \le \frac{1}{2}$

$\Rightarrow s \in \{\frac{1}{2}\}$

2. $(s_{12} = 1)$ $\frac{1}{2} \le s \le \frac{3}{2}$

$\Rightarrow s \in \{\frac{1}{2}, \frac{3}{2}\}$

	s	m
1.	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
2.	$\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$
	$\frac{3}{2}$	$-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

So the possible values of s are $\frac{1}{2}$ (degeneracy 4) and $\frac{3}{2}$ (deg. 4).

$\therefore$ The eigenvalues of H are

$$E_{s=\frac{1}{2}} = \frac{A\hbar^2}{2}\left(\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{9}{4}\right) = \frac{A\hbar^2}{2}\left(\frac{3}{4} - \frac{9}{4}\right) = -\frac{3}{4}A\hbar^2 \quad \text{(degeneracy 4)}$$

$$E_{s=\frac{3}{2}} = \frac{A\hbar^2}{2}\left(\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{9}{4}\right) = \frac{A\hbar^2}{2}\left(\frac{15}{4} - \frac{9}{4}\right) = \frac{3}{4}A\hbar^2 \quad \text{(degeneracy 4)}$$