2. *Quantum Mechanics* (Fall 2005)

Consider the one-dimensional harmonic oscillator. The Hamiltonian is

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

(a) Define the operator

$$U = e^{ipb/\hbar}$$

for some real number b. Here p is the momentum operator. What is the ground state wave function (up to the normalization constant) for the Hamiltonian

$$H = UH_0U^\dagger \quad ?$$

Note: If you know the answer, it is enough just to write it down. The derivation is allowed, but not required for full credit.

(b) Suppose a term αx^3 is added to the Hamiltonian H_0. Calculate the change in the energy of each level, through second order in α. Please write your answer as a constant independent of the level number n, times a polynomial or ratio of polynomials in n.

a) $U(b)$ is the translation operator, so given that the ground state of H_0 is $\langle x|0\rangle = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}} e^{-x^2/4x_0^2}$ where $x_0 = \frac{\hbar}{2m\omega}$

the ground state of H is

$$\langle x|0\rangle' = \langle x-b|0\rangle = \frac{1}{(2\pi)^{1/4}\sqrt{x_0}} e^{-(x-b)^2/4x_0^2}$$

$\uparrow \pm ?$

b) $H = H_0 + H' \quad H' = \alpha x^3 = \alpha x_0^3 (a^\dagger + a)^3 = \alpha x_0^3 (a^\dagger + a)(a^{\dagger 2} + a^\dagger a + aa^\dagger + a^2)$

$$= \alpha x_0^3 (a^{\dagger 3} + a^{\dagger 2}a + a^\dagger a a^\dagger + a^\dagger a^2 + aa^{\dagger 2} + aa^\dagger a + a^2 a^\dagger + a^3)$$

$$+3 \quad +1 \quad +1 \quad -1 \quad +1 \quad -1 \quad -1 \quad -3$$

Using perturbation theory

$\Delta E_n^{(1)} = \langle n|H'|n\rangle = 0$ since H' only "connects" states where $\Delta n \in \{\pm 1, \pm 3\}$ but here $\Delta n = n - n = 0$.

$$\Delta E_n^{(2)} = -\sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_m^0 - E_n^0} = -\frac{\alpha^2 x_0^6}{\hbar\omega}\left[\frac{|\langle n-3|a^3|n\rangle|^2}{n-3-n} + \frac{|\langle n-1|(a^\dagger a^2 + aa^\dagger a + a^2 a^\dagger)|n\rangle|^2}{n-1-n}\right.$$

$$\left. + \frac{|\langle n+1|(a^{\dagger 2}a + a^\dagger a a^\dagger + aa^{\dagger 2})|n\rangle|^2}{n+1-n} + \frac{|\langle n+3|a^{\dagger 3}|n\rangle|^2}{n+3-n}\right]$$

$$= -\frac{\alpha^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3\left[-\frac{1}{3}n(n-1)(n-2) - \left\{\sqrt{n}(n-1) + \sqrt{n}\,n + \sqrt{n}(n+1)\right\}^2 + \frac{1}{3}(n+1)(n+2)(n+3)\right.$$

$$\left. + \left\{\sqrt{n+1}\,n + \sqrt{n+1}(n+1) + \sqrt{n+1}(n+2)\right\}^2\right]$$

$$= -\frac{\alpha^2}{\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3\left[-\frac{1}{3}n(n-1)(n-2) - 9n^3 + (n+1)(3n+3)^2 + \frac{1}{3}(n+1)(n+2)(n+3)\right]$$