1. *Quantum Mechanics* (Fall 2005)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L}+\mathbf{s}$. Conventionally, the states are labeled $|n,l,j,m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

In parts (a) and (d) you may state the answer to lowest nonvanishing order — ignore spin-orbit and relativistic effects.

(a) If the electron is in the state $|n,l,j,m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $J_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$ $\quad J_+ \to T_1^1 \quad \Delta m \neq q = 1 \quad \Rightarrow \quad \boxed{0}$

(ii) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_+\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad \sqrt{\frac{3}{2}(\frac{3}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)} = \sqrt{\frac{15}{4} - \frac{3}{4}} = \boxed{\sqrt{3}}$

(iii) $\left\langle 2,1,\frac{1}{2},-\frac{1}{2}\middle|\mathbf{L}^2\middle|2,1,\frac{1}{2},-\frac{1}{2}\right\rangle = ?$ $\quad L^2 \to T_0^0 \quad l(l+1) = 1(1+1) = \boxed{2}$

(iv) $\left\langle 3,2,\frac{3}{2},-\frac{1}{2}\middle|\mathbf{J}^2\middle|3,2,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$ $\quad J^2 \to T_0^0 \quad j(j+1) = \frac{3}{2}(\frac{3}{2}+1) = \boxed{\frac{15}{4}}$

(v) $\left\langle 3,1,\frac{3}{2},\frac{3}{2}\middle|J_z\middle|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad J_z \to T_1^0 \quad \Delta m \neq q = 0 \quad \Rightarrow \quad \boxed{0}$

(d) What are

(i) $\left\langle 2,1,\frac{3}{2},\frac{3}{2}\middle|p_z\middle|2,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$ $\quad p_z \to T_1^0 \quad \Delta m \neq q = 0 \quad \Rightarrow \quad \boxed{0}$

(ii) $\left\langle 1,0,\frac{1}{2},\frac{1}{2}\middle|p_i p_j\middle|1,0,\frac{1}{2},\frac{1}{2}\right\rangle = ?$

a) $H: \; E_n = -\frac{1}{2}\alpha^2 mc^2 \frac{1}{n^2} \qquad L^2 = l(l+1) \qquad J^2 = j(j+1) \qquad J_z = m = m_j$

b) $n \in \mathbb{Z}^+ = \{1,2,3,\ldots\} \qquad l \in \{0,1,2,\ldots,n-1\} \qquad s = \frac{1}{2}$

$j \in \{|l-s|,\ |l-s|+1,\ |l-s|+2,\ \ldots,\ l+s-1,\ l+s\}$

$m \in \{-j, -j+1, \ldots, j-1, j\}$

c) Wigner-Eckart selection rules: $\langle \alpha' j' m' | T_k^q | \alpha j m\rangle = 0$

unless $m' = m+q$ and $|j-k| \leq j' \leq j+k$

d) ii) $p_i p_j = \underbrace{\frac{1}{3}\vec{p}^{\,2}\delta_{ij}}_{T_0^0} + \frac{1}{2}(\underbrace{\cancel{p_i p_j - p_j p_i}}_{[p_i,p_j]=0}) + \underbrace{\left[\frac{1}{2}(p_i p_j + p_j p_i) - \frac{1}{3}\vec{p}^{\,2}\delta_{ij}\right]}_{\sum_q T_2^q}$ $\quad |\frac{1}{2}-2| = \frac{3}{2} \not\leq \frac{1}{2} = j' \Rightarrow \emptyset$

$\Rightarrow \langle p_i p_j\rangle = \frac{1}{3}\langle \vec{p}^{\,2}\rangle \delta_{ij} = \frac{1}{3}2m\langle T\rangle\delta_{ij} \underset{\uparrow}{=} -\frac{2m}{3}\langle E\rangle \delta_{ij} = +\frac{1}{3}\alpha^2 m^2 c^2 \frac{1}{n^2}\delta_{ij}$

Virial theorem